1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 4

Assigned 2009 October 8
Due 2009 October 15

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

0 Conventions

The convention we're using for the continuous Fourier transform is

$$\mathcal{F}\{h\} = \widetilde{h}(f) = \int_{-\infty}^{\infty} h(t)\, e^{-i2\pi f t}\, dt \tag{0.1a}$$

$$\mathcal{F}^{-1}\left\{\widetilde{h}\right\} = h(t) = \int_{-\infty}^{\infty} \widetilde{h}(f)\, e^{i2\pi f t}\, df \tag{0.1b}$$

and for the discrete Fourier transform

$$\widehat{h}_k = \sum_{j=0}^{N-1} h_j\, e^{-i2\pi jk/N} \tag{0.2a}$$

$$h_j = \frac{1}{N} \sum_{k=0}^{N-1} \widehat{h}_k\, e^{i2\pi jk/N} \tag{0.2b}$$

1 The Forced Damped Harmonic Oscillator

This problem will show how Fourier transforms relate the impulse response (delta function driving force) and steady-state response (sinusoidal driving force) of the forced, damped harmonic oscillator of natural frequency f_0 and damping constant γ.

a) Consider the function

$$h_+(t) = \begin{cases} 0 & t < 0 \\ e^{-\gamma t} e^{i2\pi f_1 t} & t > 0 \end{cases} \tag{1.1}$$

Work out its Fourier transform $\widetilde{h}_+(f)$. (Note: this is **NOT** a delta function in f.)

b) Using the fact that $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ and your results to part a), work out the Fourier transform $\widetilde{h}_s(f)$ of

$$h_s(t) = \begin{cases} 0 & t < 0 \\ e^{-\gamma t} \sin 2\pi f_1 t & t > 0 \end{cases} \tag{1.2}$$

c) Consider the problem of a damped, driven harmonic oscillator described by the differential equation

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + (2\pi f_0)^2 x(t) = F(t)/m \ . \tag{1.3}$$

In classical mechanics we work out the response to an impulsive force; if $F(t) = p_0\delta(t - t')$, then $x(t) = p_0\, R(t - t')$ where

$$R(t - t') = \begin{cases} 0 & t < t' \\ \frac{1}{2\pi f_1 m} e^{-\gamma(t-t')} \sin 2\pi f_1 (t - t') & t > t' \end{cases} \tag{1.4}$$

where $f_1 = \sqrt{f_0^2 - (\gamma/2\pi)^2}$. Use the principle of superposition to show that the response to an arbitrary force $F(t)$ is

$$x(t) = \int_{-\infty}^{\infty} R(t - t')\, F(t')\, dt' \tag{1.5}$$

d) Work out the Fourier transform $\widetilde{R}(f)$ of the impulse response.

e) Use the convolution theorem (which you don't need to prove again) to find an expression for $\widetilde{x}(f)$ in terms of $\widetilde{R}(f)$ and $\widetilde{F}(f)$. Put in the explicit form of $\widetilde{R}(f)$ from part e).

f) Extra credit: Writing $\widetilde{R}(f)$ as a real amplitude times a phase,

$$\widetilde{R}(f) = A(f)e^{i\phi(f)} \tag{1.6}$$

work out the real functions $A(f)$ and $\phi(f)$. (Extra extra credit: plot $(2\pi f_0)^2 m A(f)$ and $\phi(f)$ versus f/f_0 for $\gamma = 2\pi f_0/10$ and $\gamma = 2\pi f_0/20$.)

2 Continuous Fourier Transforms

a) Work out the inverse Fourier transform of the frequency window

$$\widetilde{R}(f) = \begin{cases} 0 & f < -f_0 - \frac{\Delta f}{2} \\ 1 & -f_0 - \frac{\Delta f}{2} < f < +f_0 - \frac{\Delta f}{2} \\ 0 & -f_0 + \frac{\Delta f}{2} < f < f_0 - \frac{\Delta f}{2} \\ 1 & f_0 - \frac{\Delta f}{2} < f < f_0 + \frac{\Delta f}{2} \\ 0 & f > f_0 + \frac{\Delta f}{2} \end{cases} \tag{2.1}$$

b) Show that

$$\mathcal{F}\left\{ \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2} \right\} = e^{-(2\pi f)^2/2\sigma^{-2}} \tag{2.2}$$

(Hint: make a change of variables $t' = t + i\beta$ with an appropriately chosen β which completes the square so that the t dependence in the integrand is $e^{-t'^2/2\sigma^2}$. Since the limits of integration in t' are $-\infty + i\beta$ to $\infty + i\beta$ you need to explain why the extra contributions to the integral as t' goes from $-\infty + i\beta$ to $-\infty$ and ∞ to $\infty + i\beta$ vanish.)

3 Discrete Fourier Transforms

a) Use

$$\sum_{k=0}^{N-1} e^{i2\pi(j-\ell)k/N} = N\,\delta_{j,\ell \bmod N} \tag{3.1}$$

to show that

$$\sum_{j=0}^{N-1} g_j^*\, h_j = \frac{1}{N} \sum_{k=0}^{N-1} \widehat{g}_k^*\, \widehat{h}_k \tag{3.2}$$

b) Consider the 3 Hz sine wave $h(t) = \sin(2\pi[3\,\mathrm{Hz}]t)$, sampled at $\delta t = 0.25\,\mathrm{s}$. Sketch the continuous function $h(t)$ from $t = 0$ to $t = 2.0\,\mathrm{s}$, and put dots at the samples $h_j = h(j\,\delta t)$.

c) $\{h_j\}$ is also the discretization of a lower-frequency sinusoidal function, i.e., $h_j = \mathfrak{h}(j\delta t)$. What is $\mathfrak{h}(t)$?

d) What are the Fourier components $\{\widehat{h}_k | k = 0, \ldots, 7\}$? What about $\{\widehat{h}_k | k = -4, \ldots, 3\}$?