1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 5

Assigned 2009 October 15
Due 2009 October 22

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Colored Noise

This problem should be done in your favorite numerical analysis environment. Please turn in a printout of the final set of commands you used as well as the plots themselves.

Consider a $T = 16\,\text{sec}$ of data sampled at $\frac{1}{\delta t} = 1024\,\text{Hz}$, so that $N = 16\,384$.

a) Generate an N-point series $\{x_j\}$ of random samples drawn from a distribution with zero mean and unit variance, i.e., $\langle x_j \rangle = 0$ and $\langle x_j x_\ell \rangle = \delta_{j\ell}$. (This can be done in matplotlib with `randn(N)`.) Plot x_j versus $t_j - t_0 = j\delta t$ for $t_j - t_0$ from 0 to 16 sec.

b) Take the discrete Fourier transform $\widehat{x}_k$ and plot $|\widehat{x}_k|$ versus $f_k = k\delta f = k/T$ for f_k from $-512\,\text{Hz}$ to $512\,\text{Hz}$.

c) Calculate the average

$$\frac{1}{N} \sum_{k=-N/2}^{N/2-1} |\widehat{x}_k|^2 \tag{1.1}$$

How does it compare to your theoretical expectations for $\left\langle |\widehat{x}_k|^2 \right\rangle$?

d) Construct $\widehat{y}_k = \widehat{x}_k \, e^{-f_k^2/2\sigma_f^2}$, where $\sigma_f = 16\,\text{Hz}$, for $k \in [-N/2, N/2-1]$ and plot $|\widehat{y}_k|$ versus f_k for f_k from $-512\,\text{Hz}$ to $512\,\text{Hz}$.

e) Take the inverse discrete Fourier Transform to get y_j and plot y_j versus $t_j - t_0$ for $t_j - t_0$ from 0 to 16 sec.

2 Power Spectral Density

Consider a single random variable ψ which is equally likely to fall anywhere between 0 and 2π, so that expectation values of random variables whose only randomness comes from ψ can be calculated as

$$\langle F(\psi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} F(\psi)\, d\psi \ . \tag{2.1}$$

Let $x(t) = A\cos(2\pi f_0 t + \psi)$ for some fixed A and f_0.

a) Find $\langle x(t)\rangle$ and $\langle x(t)x(t')\rangle$ and show that $x(t)$ is wide-sense stationary.

b) Find the power spectral density $P_x(f)$.

3 Averages and Expectation Values

Let $\{x_k|k = 0\ldots N-1\}$ be a set of N uncorrelated random variables all drawn from the same distribution with (possibly unknown) mean μ and variance σ^2 so that the expectation values are

$$\langle x_k\rangle = \mu \tag{3.1a}$$

$$\langle (x_k-\mu)(x_\ell-\mu)\rangle = \delta_{k\ell}\,\sigma^2 \tag{3.1b}$$

a) Consider the average of the N instantiations

$$\overline{x} = \frac{1}{N}\sum_{k=0}^{N-1} x_k \tag{3.2}$$

and show that its expectation value $\langle\overline{x}\rangle$ is equal to μ. (This is pretty easy to show if you remember that the expectation value is a linear operation, so that $\langle\alpha+\beta\rangle = \langle\alpha\rangle+\langle\beta\rangle$.) This means that $\overline{x}$ is an *unbiased estimator* of the mean of the underlying distribution, even though it's constructed from a finite number of samples from that distribution.

b) Calculate the expected variance of $\overline{x}$, i.e.,

$$\left\langle (\overline{x}-\langle\overline{x}\rangle)^2\right\rangle = \left\langle (\overline{x}-\mu)^2\right\rangle = \text{ what?} \tag{3.3}$$

c) Suppose we know the exact value of μ (via some sort of physical principle or something); we could use

$$\overline{(x-\mu)^2} = \frac{1}{N}\sum_{k=0}^{N-1}(x_k-\mu)^2 \tag{3.4}$$

as an estimator of the underlying variance σ^2. Show that this is an unbiased estimator, i.e., that its expectation value is indeed σ^2:

$$\left\langle \overline{(x-\mu)^2}\right\rangle = \sigma^2 \tag{3.5}$$

d) Now suppose the true mean $\mu = \langle x \rangle$ is not known, and must also be estimated from the same N data points. We can consider

$$\overline{(x - \overline{x})^2} = \frac{1}{N} \sum_{k=0}^{N-1} (x_k - \overline{x})^2 \tag{3.6}$$

as a potential estimator of σ^2. Calculate its expectation value

$$\left\langle \overline{(x - \overline{x})^2} \right\rangle \ , \tag{3.7}$$

keeping in mind that both x_k and $\overline{x}$ are random variables. This will *not* be equal to σ^2 which means (3.6) is a *biased* estimator.

e) Modify (3.6) to produce an unbiased estimator of σ which can be calculated from only the samples $\{x_k\}$. (This can't include the unknown actual value of μ nor any expectation values, only averages calculated from the actual samples $\{x_k\}$.)