1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 6

Assigned 2009 October 22
Due 2009 October 29

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1.1 Windowing in PSD Estimates

(Note: In this problem you should repeat analogous calculations from the class notes as necessary, rather than just quoting the results.) The problem of spectral leakage can be reduced by multiplying the time series $\{h_j = h(t_0 + j\,\delta t)\}$ by a window function $\{w_j\}$ before performing the discrete Fourier transform:

$$\widehat{h}^w_k = \sum_{j=0}^{N-1} w_j\, h_j\, e^{-i2\pi jk/N} \ . \tag{1.1}$$

a) Use the inverse Fourier transform

$$h(t) = \int_{-\infty}^{\infty} \widetilde{h}(f)\, e^{i2\pi f(t-t_0)}\, df \tag{1.2}$$

to find an expression for discrete Fourier transform of the windowed data of the form

$$\widehat{h}^w_k = \int_{-\infty}^{\infty} \Lambda([f_k - f]\delta t)\, \widetilde{h}(f)\, df \tag{1.3}$$

where $\Lambda(x)$ is some function constructed using the N-point window $\{w_j\}$, which you will determine.

b) Use (1.3) to find the expectation value $\left\langle \left| \widehat{h}^w_k \right|^2 \right\rangle$ in terms of the PSD $S_h(f)$ and the function $\Lambda(x)$.

c) Show that for integer j and ℓ,

$$\int_{-1/2}^{1/2} e^{-i2\pi(j-\ell)x}\,dx = \delta_{j\ell} \tag{1.4}$$

d) Use (1.4) to evaluate

$$\int_{-1/2}^{1/2} |\Lambda(x)|^2\,dx \tag{1.5}$$

in terms of the mean square window value

$$\overline{w^2} = \frac{1}{N}\sum_{j=0}^{N-1}(w_j)^2 \tag{1.6}$$

(Extra credit: use your result to show that for $\Delta_N(x) = \sum_{j=0}^{N-1} e^{-i2\pi jx}$ considered in class, $\int_{-1/2}^{1/2} |\Delta_N(x)|^2\,dx = N$.)

e) Assuming that $|\Lambda(x)|^2$ is sharply enough peaked to be treated as an approximate Dirac delta function

$$|\Lambda(x)|^2 \approx \mathcal{A}\,\delta(x) \tag{1.7}$$

find the proportionality constant $\mathcal{A}$.

f) Using the approximation (1.7) (with the explicit value you found for $\mathcal{A}$), write an approximate expression for the expectation value $\left\langle \left|\widehat{h}_k^w\right|^2 \right\rangle$ in terms of $S_h(f_k)$, δt, N, and $\overline{w^2}$.

g) Use the result of part f) to write a windowed periodogram P_k^w constructed from $\left|\widehat{h}_k^w\right|^2$ such that $\langle P_k^w\rangle \approx S_h(f_k)$.

1.2 Hann Window

Consider the window function

$$w_j = \frac{1}{2}\left(1 - \cos\frac{2\pi j}{N}\right) \tag{1.8}$$

a) Analytically calculate the explicit form of $\Lambda(x)$ and $|\Lambda(x)|^2$

b) For $N = 64$, plot w_j versus j.

c) Again for $N = 64$, plot $|\Lambda(x)|^2$ and $|\Delta_N(x)|^2$ versus x for $-1/8 < x < 1/8$. Make the plot both for a linear vertical scale and for a logarithmic scale from 1 to 10^4. (In matplotlib and matlab, the function `semilogy()` can be used to make plots with a linear horizontal and logarithmic vertical scale.)

2 Exercises in Logic and Probability

Do all of the problems at the end of Chapter 2 of Gregory.