1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 7

Assigned 2009 October 29
Due 2009 November 5

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

0 Useful Integrals

The following integrals are likely to be useful in this problem set:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}} \tag{0.1a}$$

$$\int_{-\infty}^{\infty} x\,e^{-\alpha x^2}\,dx = 0 \tag{0.1b}$$

$$\int_{-\infty}^{\infty} x^2\,e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \tag{0.1c}$$

$$\int_{-\infty}^{\infty} x^3\,e^{-\alpha x^2}\,dx = 0 \tag{0.1d}$$

$$\int_{-\infty}^{\infty} x^4\,e^{-\alpha x^2}\,dx = \frac{3}{4\alpha^2}\sqrt{\frac{\pi}{\alpha}} \tag{0.1e}$$

(Extra credit: show that each of these is true.)

Also, the following definition and identity will be useful

$$\operatorname{erfc}(v) = \frac{2}{\sqrt{\pi}}\int_{v}^{\infty} e^{-u^2}\,du = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{-v} e^{-u^2}\,du \tag{0.2}$$

1 Gaussian Random Data

Consider a particular Fourier component $\widehat{h}_k$ of a random data series, and, writing $\widehat{h}_k = \xi + i\eta$, let its real and imaginary parts be random variables whose joint pdf is

$$p(\widehat{h}_k) = p(\xi,\eta) = \frac{\exp(-\xi^2/S_k)}{\sqrt{\pi S_k}} \frac{\exp(-\eta^2/S_k)}{\sqrt{\pi S_k}} = \frac{\exp(-|\widehat{h}_k|^2/S_k)}{\pi S_k} \tag{1.1}$$

so that the expectation value of any function of $\widehat{h}_k$ is

$$\left\langle F(\widehat{h}_k) \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi + i\eta)\, p(\xi,\eta)\, d\xi\, d\eta \tag{1.2}$$

a) Show that

$$\left\langle \left|\widehat{h}_k\right|^2 \right\rangle = S_k \tag{1.3}$$

b) Show that

$$\left\langle \left|\widehat{h}_k\right|^4 \right\rangle = 2S_k^2 \tag{1.4}$$

(We assumed this in class to show that the expected standard deviation of the periodogram $P_k = \frac{\delta t}{N} \left|\widehat{h}_k\right|^2$ was equal to its expected mean.)

c) Consider the case where the original data stream is complex, so that the real and imaginary parts of $\widehat{h}_k$ for $k = -N/2, \ldots, N/2-1$ are independent random variables. Write the joint pdf

$$p(\{\widehat{h}_k\}) = \prod_{k=-N/2}^{N/2-1} p(\widehat{h}_k) \tag{1.5}$$

in the form

$$p(\{\widehat{h}_k\}) = \mathcal{A} \exp(L(\{\widehat{h}_k\})) \tag{1.6}$$

where $\mathcal{A}$ is a normalization constant independent of the Fourier components, and $L(\{\widehat{h}_k\})$ is some explicit expression in terms of the $\{\widehat{h}_k\}$ and $\{S_k\}$.

d) Use the identifications

$$\widehat{h}_k \sim (\delta t)^{-1}\, \widetilde{h}(f_k) \tag{1.7a}$$

$$\left\langle \left|\widehat{h}_k\right|^2 \right\rangle \sim \frac{N}{\delta t} S_h(f_k) \tag{1.7b}$$

to write $L(\{\widehat{h}_k\}) \sim L[\widetilde{h}]$ in the limit that δt and δf go to zero as a definite integral over f. (There should be no reference to anything discrete in your answer.) You have thereby worked out the pdf

$$p[\widetilde{h}] = \mathcal{A} \exp(L[\widetilde{h}]) \tag{1.8}$$

up to an overall normalization $\mathcal{A}$.

e) Extra credit: repeat the process for the case where $\{\widehat{h}_k\}$ are the Fourier components associated with a real time series; you may find it useful to make reference to the one-sided PSD $S_h^{\text{1-sided}}(f)$ defined in class. (See the lecture notes on Fourier methods.)

2 Upper Limits

Consider an experiment designed to measure an unknown physical quantity x, which returns a value y whose pdf is defined by the likelihood function

$$p(y|x) = \frac{e^{-(y-x)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \tag{2.1}$$

a) Suppose the experiment has been performed and the result $\hat{y}$ has been found. Calculate the frequentist upper limit $x_{\text{UL}}^{\text{freq}}$ at confidence level α, defined by

$$\int_{\hat{y}}^{\infty} p(y|x_{\text{UL}}^{\text{freq}})\,dy = \alpha\ . \tag{2.2}$$

You should be able to write this with the help of the inverse complementary error function $\operatorname{erfc}^{-1}(\xi)$. Note that $\operatorname{erfc}^{-1}(\xi)$ is positive if $0 < \xi < 1$ and negative if $1 < \xi < 2$, and that $\operatorname{erfc}^{-1}(2-\xi) = -\operatorname{erfc}^{-1}(\xi)$

b) Consider a Bayesian analysis with a uniform prior on x, so that by Bayes's theorem, the posterior is

$$p(x|y) = \frac{p(x)}{p(y)}\,p(y|x) = \mathcal{A}\,p(y|x)\ . \tag{2.3}$$

Using the explicit form of the likelihood (2.1) and the normalization requirement

$$\int_{-\infty}^{\infty} p(x|y)\,dx = 1 \tag{2.4}$$

find the value of $\mathcal{A}$ and therefore the explicit form of the posterior $p(x|y)$.

c) Supposing again that we've performed the experiment and found a result $\hat{y}$, find the Bayesian upper limit $x_{\text{UL}}^{\text{Bayes}}$ at confidence level α, defined by

$$\int_{-\infty}^{x_{\text{UL}}^{\text{Bayes}}} p(x|\hat{y})\,dx = \alpha \tag{2.5}$$

d) For the case where $\alpha = 0.9$, write $x_{\text{UL}}^{\text{freq}}$ and $x_{\text{UL}}^{\text{Bayes}}$ explicitly in terms of $\hat{y}$ and σ, with any constants evaluated to two significant figures. (You'll need to refer to the explicit value of $\operatorname{erfc}^{-1}(\xi)$ for a particular ξ; in matplotlib you can get access to the inverse complementary error function via `from scipy.special import erfcinv`.)

e) Suppose now that x is physically constrained to be positive and let the prior be uniform for positive x, so that the posterior can be written in terms of the Heaviside step function

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \tag{2.6}$$

as

$$p(x|y) = \frac{p(x)}{p(y)}\, p(y|x) = \mathcal{B}\,\Theta(x)\,p(y|x)\ . \tag{2.7}$$

Use the normalization condition

$$1 = \int_0^\infty p(x|y)\,dx = \mathcal{B}\int_0^\infty p(y|x)\,dx \tag{2.8}$$

to find the value of $\mathcal{B}$ and therefore the explicit form of $p(x|y)$.

f) Supposing again that we've performed the experiment and found a result $\hat{y}$, calculate the Bayesian upper limit $x_{\text{UL}}^{\text{Bayes+}}$ associated with the posterior (2.7), defined by

$$\int_0^{x_{\text{UL}}^{\text{Bayes+}}} p(x|\hat{y})\,dx = \alpha \tag{2.9}$$

3 Change of Variables

a) Define traditional spherical coördinates (θ, ϕ), with θ being the angle down from the zenith (so that $\theta = 0$ is the zenith and $\theta = \pi/2$ is the horizon) and ϕ being an azimuthal angle which runs from 0 to 2π. Consider an event which is equally likely to occur anywhere in the visible half of the sky (i.e., above the horizon). What is the pdf $p(\theta, \phi)$ for its direction of origin? (Note that this is *not* a constant, and should be normalized so that $\int_0^{2\pi}\int_0^{\pi/2} p(\theta,\phi)\,d\theta\,d\phi = 1$.)

b) Let

$$n_x = \sin\theta\cos\phi \tag{3.1a}$$
$$n_y = \sin\theta\sin\phi \tag{3.1b}$$

be the projections onto two horizontal directions of the unit vector pointing to the event. Find the joint pdf $p(n_x, n_y)$ for those two variables. (Your answer should not contain θ or ϕ, although they may be convenient for intermediate steps.)

c) What is the region of the (n_x, n_y) plane which corresponds to $0 \le \theta \le \pi/2$, $0 \le \phi < 2\pi$?

d) Extra credit: marginalize over n_y to find $p(n_x)$.