Probability
(Devore Chapter Two)

1016-351-03: Probability*

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Contents

Tuesday 8 December 2009

1 Axiomatic Probability

Many of the rules of probability appear to be self-evident, but an important part of mathematics is illustrating that our intuition agrees with the logical outcome of our models. To that end, Devore develops with some care a mathematical theory of probability. We'll mostly summarize the key definitions and results here.

1.1 Outcomes and Events

Devore defines probability in terms of an **experiment** which can have one of a set of possible **outcomes**.

- The **sample space** of an experiment, written $\mathcal{S}$, is the set of all possible outcomes.
- An **event** is a subset of $\mathcal{S}$, a set of possible outcomes to the experiment. Special cases are:
  - The **null event** $\varnothing$ is an event consisting of no outcomes (the empty set)
  - A **simple event** consists of exactly one outcome
  - A **compound event** consists of more than one outcome

The sample space $\mathcal{S}$ itself an event, of course.

One example of an experiment is flipping a coin three times. The outcomes in that case are HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT. Possible outcomes include:

- Exactly two heads: $\{HHT, HTH, THH\}$
- The first flip is heads: $\{HHH, HHT, HTH, HTT\}$
- The second and third flips are the same: $\{HHH, HTT, THH, TTT\}$

Another example is a game of craps, in which:

- if a 2, 3 or 12 is rolled on the first roll, the shooter loses
- if a 7 or 11 is rolled on the first roll, the shooter wins
- if a 4, 5, 6, 8, 9, or 10 is rolled on the first roll, the dice are rolled again until the either that number or a 7 comes up, in which case the shooter wins or loses, respectively.

In this case there are an infinite number of outcomes in $\mathcal{S}$, some of which are: 2, 3, 7, 11, 12, 4|4, 4|7, 5|5, 5|7, 6|6, 6|7, 8|8, 8|7, 9|9, 9|7, 10|10, 10|7, 4|2|4, 4|3|4, 4|5|4, 4|6|4, ... Possible events include: the shooter wins $\{7, 11, 4|4, 5|5, 6|6, 8|8, \ldots\}$; the shooter loses $\{2, 3, 12, 4|7, \ldots\}$; the dice are thrown exactly once $\{2, 3, 7, 11, 12\}$, etc.

Since an event is a set of outcomes, we can use all of the machinery of set theory, specifically:

- The **complement** A' of a set A, is the set of all outcomes in $\mathcal{S}$ which are *not* in A.
- The **union** $A \cup B$ of two sets A and B, is the set of all outcomes which are in A *or* B, including those which are in both.
- The **intersection** $A \cap B$ is the set of all outcomes which are in both A *and* B.

In the case of coin flips, if the events are $A = \{HHT, HTH, THH\}$ (exactly two heads) and $B = \{HHH, HHT, HTH, HTT\}$ (first flip heads), we

$$A' = \{HHH, HTT, THT, TTH, TTT\}$$
$$A \cup B = \{HHH, HHT, HTH, HTT, THH\}$$
$$A \cap B = \{HHT, HTH\}$$

Another useful definition is that A and B are **disjoint** or **mutually exclusive** events if $A \cap B = \varnothing$.

1.2 Rules of Probability

Having formally defined what we mean by an event, we can proceed to define the probability of that event, which we think of as the chance that it will occur. Devore starts with three axioms

1. For any event A, $P(A) \geq 0$
2. $P(\mathcal{S}) = 1$
3. Given an infinite collection A_1, A_2, A_3, $\ldots$ of *disjoint* events,

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \tag{1.1}$$

From there he manages to derive a bunch of other sensible results, such as

1. For any event A, $P(A) \leq 1$
2. $P(\varnothing) = 0$
3. $P(A') = 1 - P(A)$

One useful result concerns the probability of the union of any two events. Since $A \cup B = (A \cap B') \cup (A \cap B) \cup (A' \cap B)$, the union of three disjoint events,

$$P(A \cup B) = P(A \cap B') + P(A \cap B) + P(A' \cap B) \tag{1.2}$$

On the other hand, $A = (A \cap B') \cup (A \cap B)$ and $B = (A \cap B) \cup (A' \cap B)$, so

$$P(A) = P(A \cap B') + P(A \cap B) \tag{1.3a}$$
$$P(B) = P(A \cap B) + P(A' \cap B) \tag{1.3b}$$

which means that

$$P(A) + P(B) = P(A \cap B') + 2P(A \cap B) + P(A' \cap B) = P(A \cup B) + P(A \cap B) \tag{1.4}$$

so

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{1.5}$$

1.3 Assigning Probabilities

The axioms of probability let us relate the probabilities of different events, but they don't tell us what those probabilities should be in the first place. If we have a way of assigning probabilities to each outcome, and therefore each simple event, then we can use the sum rule for disjoint events to write the probability of any event as the sum of the probabilities of the simple events which make it up. I.e.,

$$P(A) = \sum_{E_i \text{ in } A} P(E_i) \tag{1.6}$$

One possibility is that each outcome, i.e., each simple event might be equally likely. In that case, if there are N outcomes total, the probability of each of the simple events is $P(E_i) = 1/N$ (so that $\sum_{i=1}^{N} P(E_i) = P(\mathcal{S}) = 1$), and in that case

$$P(A) = \sum_{E_i \text{ in } A} \frac{1}{N} = \frac{N(A)}{N} \tag{1.7}$$

where $N(A)$ is the number of outcomes which make up the event A.

Note, however, that one has to consider whether it's appropriate to take all of the outcomes to be equally likely. For instance, in our craps example, we considered each roll, e.g., 2 and 4, to be its own outcome. But you can also consider the rolls of the individual dice, and then the two dice totalling 4 would be a composite event consisting of the outcomes $(1, 3)$, $(2, 2)$, and $(3, 1)$. For a pair of fair dice, the 36 possible outcomes defined by the numbers on the two dice taken in order (suppose one die is green and the other red) are equally likely outcomes.

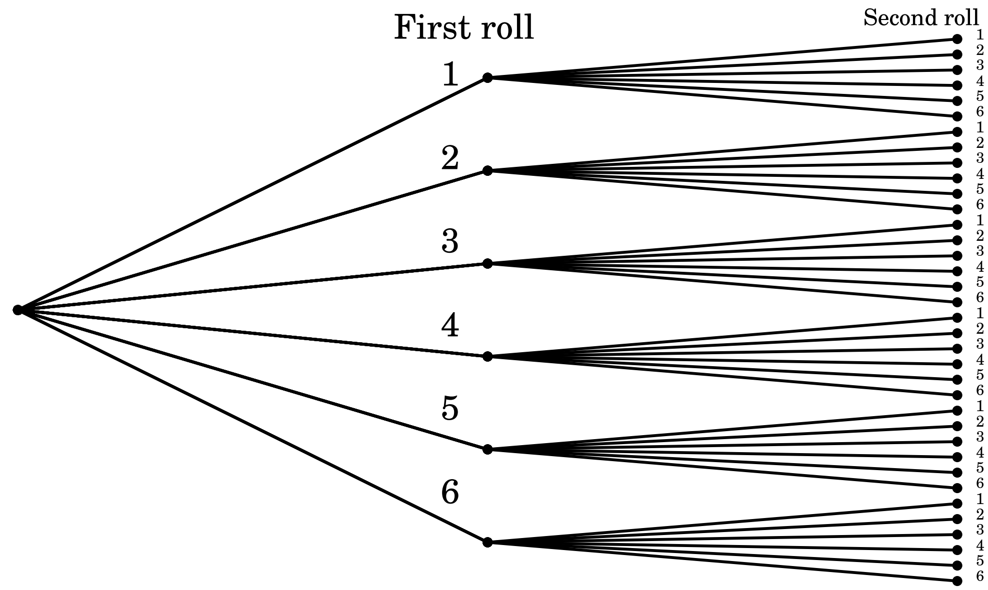
2 Counting Techniques

2.1 Ordered Sequences

We can come up with 36 as the number of possible results on a pair of fair dice in a couple of ways. We could make a table

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

which is also useful for counting the number of occurrences of each total. Or we could use something called a tree diagram:



This works well for counting a small number of possible outcomes, but already with 36 outcomes it is becoming unwieldy. So instead of literally counting the possible outcomes, we should calculate how many there will be. In this case, where the outcome is an ordered pair of numbers from 1 to 6, there are 6 possibilities for the first number, and corresponding to each of those there are 6 possibilities for the second number. So the total is $6 \times 6 = 36$.

More generally, if we have an ordered set of k objects, with n_1 possibilities for the first, n_2 for the second, etc, the number of possible ordered k-tuples is $n_1 n_2 \ldots n_k$, which we can also write as

$$\prod_{i=1}^{k} n_k \ . \tag{2.1}$$

2.2 Permutations and Combinations

Consider the probability of getting a poker hand (5 cards out of the 52-card deck) which consists entirely of hearts.[1] Since there are four different suits, you might think the odds are $(1/4)(1/4)(1/4)(1/4)(1/4) = (1/4)^5 = 1/4^5$.

[1]This is, hopefully self-apparently, one-quarter of the probability of getting a flush of any kind.

However, once a heart has been drawn on the first card, there are only 12 hearts left in the deck out of 51; after two hearts there are 11 out of 50, etc., so the actual odds are

$$P(\heartsuit\heartsuit\heartsuit\heartsuit\heartsuit) = \left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\left(\frac{10}{49}\right)\left(\frac{9}{48}\right) \tag{2.2}$$

This turns out not to be the most effective way to calculate the odds of poker hands, though. (For instance, it's basically impossible to do a card-by-card accounting of the probabability of getting a full house.) Instead we'd like to take the approach of counting the total number of possible five-card hands (outcomes) and then counting up how many fall into a particular category (event). The terms for the quantities we will be interested in are **permutation** and **combination**.

First, let's consider the number of possible sequences of five cards drawn out of a deck of 52. This is the permutation number of permutations of 5 objects out of 52, called $P_{5,52}$. The first card can be any of the 52; the second can be any of the remaining 51; the third can be any of the remaining 50, etc. The number of permutations is

$$P_{5,52} = 52 \times 51 \times 50 \times 49 \times 48 \tag{2.3}$$

In general

$$P_{k,n} = n(n-1)(n-2)\cdots(n-k+1) = \prod_{\ell=0}^{k-1}(n-\ell)\ . \tag{2.4}$$

Now, there is a handy way to write this in terms of the factorial function. Remember that the factorial is defined as

$$n! = n(n-1)(n-1)\cdots(2)(1) = \prod_{\ell=1}^{n}\ell \tag{2.5}$$

with the special case that $0! = 1$. Then we can see that

$$\begin{aligned}\frac{n!}{(n-k)!} &= \frac{n(n-1)(n-2)\cdots(n-k+1)\cancel{(n-k)}\cancel{(n-k-1)}\cdots\cancel{(2)}\cancel{(1)}}{\cancel{(n-k)}\cancel{(n-k-1)}\cdots\cancel{(2)}\cancel{(1)}} \\ &= P_{k,n}\end{aligned} \tag{2.6}$$

Note in particular that the number of ways of arranging n items is

$$P_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \tag{2.7}$$

Now, when we think about the number of different poker hands, actually we don't consider the cards in a hand to be ordered. So in fact all we care about is the number of ways of choosing 5 objects out of a set of 52, without regard to order. This is the number of **combinations**, which is sometimes written $C_{5,52}$, but which we'll write as $\binom{52}{5}$, pronounced "52 choose 5". When we counted the number of different permutations of 5 cards out of 52, we actually counted each possible hand a bunch of times, once for each of the ways of arranging the cards. There are $P_{5,5} = 5!$ different ways of arranging the five cards of a poker hand, so the number of permutations of 5 cards out of 52 is the number of combinations times the number of permutations of the 5 cards among themselves:

$$P_{5,52} = \binom{52}{5} P_{5,5} \tag{2.8}$$

The factor of $P_{5,5} = 5!$ is the factor by which we overcounted, so we divide by it to get

$$\binom{52}{5} = \frac{P_{5,52}}{P_{5,5}} = \frac{52!}{47!5!} = 2598960 \tag{2.9}$$

or in general

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{2.10}$$

So to return to the question of the odds of getting five hearts, there are $\binom{52}{5}$ different poker hands, and $\binom{13}{5}$ different hands of all hearts (since there are 13 hearts in the deck), which means the probability of the event $A = \heartsuit\heartsuit\heartsuit\heartsuit\heartsuit$ is

$$P(A) = \frac{N(A)}{N} = \frac{\binom{13}{5}}{\binom{52}{5}} = \frac{\frac{13!}{8!5!}}{\frac{52!}{47!5!}} = \frac{13!47!}{8!52!} = \frac{(13)(12)(11)(10)(9)}{(52)(51)(50)(49)(48)} \tag{2.11}$$

which is of course what we calculated before. Numerically, $P(A) \approx 4.95 \times 10^{-4}$, while $1/4^5 \approx 9.77 \times 10^{-4}$. The odds of getting any flush are four times the odds of getting an all heart flush, i.e., 1.98×10^{-3}.

Actually, if we want to calculate the odds of getting a flush, we have over-counted somewhat, since we have also included straight flushes, e.g., 4♡-5♡-6♡-7♡-8♡. If we want to count only hands which are flushes, we need to subtract those. Since aces can count as either high or low, there are ten different all-heart straight flushes, which means the number of different all-heart flushes which are not straight flushes is

$$\binom{13}{5} - 10 = \frac{13!}{8!5!} - 10 = 1287 - 10 = 1277 \tag{2.12}$$

and the probability of getting an all-heart flush is 4.92×10^{-4}, or 1.97×10^{-3} for any flush.

Exercise: work out the number of possible straights and therefore the odds of getting a straight.

Practice Problems

2.5, 2.9, 2.13, 2.17, 2.29, 2.33, 2.43

Thursday 10 December 2009

3 Conditional Probabilities and Tree Diagrams

3.1 Example: Odds of Winning at Craps

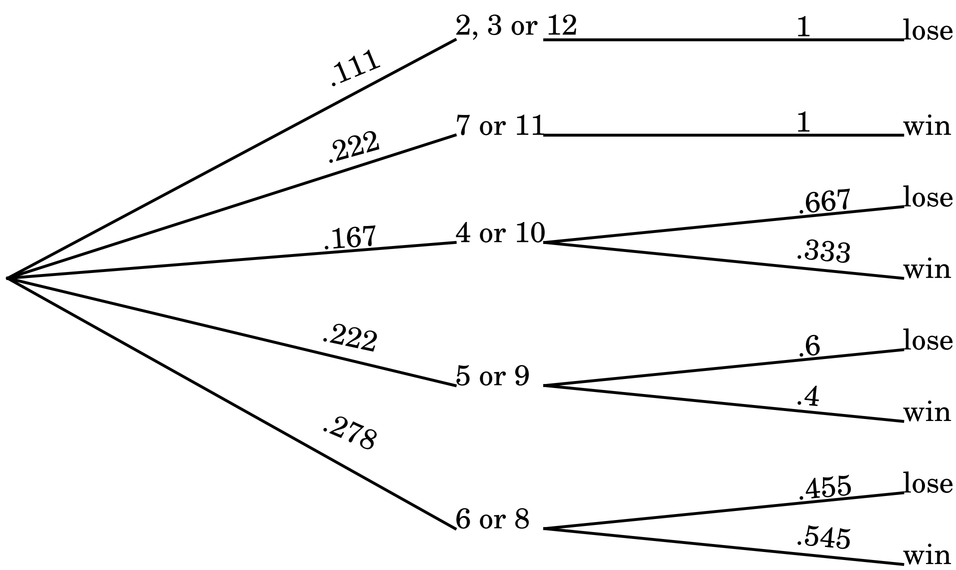
Although there are an infinite number of possible outcomes to a craps game, we can still calculate the probability of winning.

First, the sample space can be divided up into mutually exclusive events based on the result of the first roll:

Event	Probability	Result of game
2, 3 or 12 on 1st roll	$\frac{1+2+1}{36} = \frac{4}{36} \approx 11.1\%$	lose
7 or 11 on 1st roll	$\frac{6+2}{36} = \frac{8}{36} \approx 22.2\%$	win
4 or 10 on 1st roll	$\frac{3+3}{36} = \frac{6}{36} \approx 16.7\%$	???
5 or 9 on 1st roll	$\frac{4+4}{36} = \frac{8}{36} \approx 22.2\%$	???
6 or 8 on 1st roll	$\frac{5+5}{36} = \frac{10}{36} \approx 27.8\%$	???

The last three events each contain some outcomes that correspond to winning, and some that correspond to losing. We can figure out the probability of winning if, for example, you roll a 4 initially. Then you will win if another 4 comes up before a 7, and lose if a 7 comes up before a 4. On any given roll, a 7 is twice as likely to come up as a 4 (6/36 vs 3/36), so the odds are $6/9 = 2/3 \approx 66.7\%$ that you will roll a 7 before a 4 and lose. Thus the odds of losing after starting with a 4 are 66.7%, while the odds of winning after starting with a 4 are 33.3%. The same calculation applies if you get a 10 on the first roll. This means that the $6/36 \approx 16.7\%$ probability of rolling a 4 or 10 initially can be divided up into a $4/36 \approx 11.1\%$ probability to start with a 4 or 10 and eventually lose, and a $2/36 \approx 5.6\%$ probability to start with a 4 or 10 and eventually win.

We can summarize this branching of probabilities with a tree diagram:



The probability of winning given that you've rolled a 4 or 10 initially is an example of a conditional probability. If A is the event "roll a 4 or 10 initially" and B is the event "win the game", we write the conditional probability for event B given that A occurs as $P(B|A)$. We have argued that the probability for both A and B to occur, $P(A \cap B)$, should be the probability of A times the conditional probability of B given A, i.e.,

$$P(A \cap B) = P(B|A)P(A) \tag{3.1}$$

We can use this to fill out a table of probabilities for different sets of outcomes of a craps game, analogous to the tree diagram.

A	$P(A)$	B	$P(B\|A)$	$P(A \cap B) = P(B\|A)P(A)$
2, 3 or 12 on 1st roll	.111	lose	1	.111
7 or 11 on 1st roll	.222	win	1	.222
4 or 10 on 1st roll	.167	lose	.667	.111
		win	.333	.056
5 or 9 on 1st roll	.222	lose	.6	.133
		win	.4	.089
6 or 8 on 1st roll	.278	lose	.545	.152
		win	.455	.126

Since the rows all describe disjoint events whose union is the sample space $\mathcal{S}$, we can add the probabilities of winning and find that

$$P(\text{win}) \approx .222 + .056 + .089 + .126 \approx .493 \tag{3.2}$$

and

$$P(\text{lose}) \approx .111 + .111 + .133 + .152 \approx .507 \tag{3.3}$$

3.2 Definition of Conditional Probability

We've motivated the concept of conditional probability and applied it via (3.1). In fact, from a formal point of view, conditional probability is *defined* as

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \ . \tag{3.4}$$

We actually used that definition in another context above without realizing it, when we were calculating the probability of rolling a 7 before rolling a 4. We know that $P(7) = 6/36$ and $P(4) = 3/36$ on any given roll. The probability of rolling a 7 given that the game ends on that throw is

$$P(7|7 \cup 4) = \frac{P(7)}{P(7 \cup 4)} = \frac{P(7)}{P(7) + P(4)} = \frac{6/36}{9/36} = \frac{6}{9} \tag{3.5}$$

We calculated that using the definition of conditional probability.

3.3 The Monty Hall Problem

This is a classic problem in logic and probability, usually described in terms of the game show *Let's Make a Deal*.

You are given a choice of three doors; behind one there is a valuable prize (a new car), and behind the other two are booby prizes (goats). The car and goats were randomly placed before the game, and there is nothing special about any of the doors. You choose door #1. Before you open it, the host, Monty Hall, opens one of the other two doors, reveals that there is a goat behind it. You are then given the opportunity to switch from door #1 to the other unopened door.

Monty was obligated to open one of the doors, knows which door has the car behind it, and deliberately chose a door with a goat.

- What is the probability that you will win the car if you switch? What is the probability that you will win if you don't switch?
- Suppose now that Monty did not know where the car was, and chose one of the two unopened doors at random. Given that that door happened to contain a goat, what is the probability that you will win if you switch? If you don't switch?

This problem (which has caused vehement disagreements among very intelligent and educated people), can be attacked using the tools of tree diagrams and conditional probabilities.

Practice Problems

2.45, 2.59, 2.63, 2.71, 2.105 parts a & b

Tuesday 15 December 2009

Guest lecture by Dr. Chulmin Kim; see Dr. Kim's notes and worksheet

4 Bayes's Theorem