## Dr. Kim's Note (December 17th)

The values taken on by the random variable *X* are random, but the values follow the pattern given in the random variable table. What is a typical value of a random variable *X*? The solution is given by the following definition:

**Mean of a Discrete Random variable**

Suppose that *X* is a discrete random variable whose distribution is

| Value of X | Probability |
|---|---|
| $x_1$ | $p_1$ |
| $x_2$ | $p_2$ |
| : | : |
| $x_k$ | $p_k$ |

To find the **mean** of X, multiply each possible value by its probability, then add all the products: $\mu_X = x_1 p_1 + x_2 p_2 + \cdots + x_k p_k = \sum_{i=1}^{k} x_i p_i$ .

This means that the average or expected value, $\mu_X$ of the random variable $X$ is equal to the sum of all possible values of the variable, $x_i$'s, multiplied by the probabilities of each value happening.

In our 2 tosses of a coin example, we can compute the average number of heads in 2 tosses by 0(1/4)+1(1/2)+2(1/4)=1. That is, the average number or expected number of heads in 2 tosses is one head.

A more helpful way to implement this formula is to create the random variable table again, but now add an additional column to the table, and call it $X*P(x)$. In this third column multiply the value of $X$ by the probability. For example,

| $X$ | $P(x)$ | $X*P(x)$ |
|---|---|---|
| 0 | 1/4 | 0 |
| 1 | 1/2 | 1/2 |
| 2 | 1/4 | 1/2 |

then, the average or expected value of $X$ is found by adding up all the values in the third column to obtain $\mu_X = 1$.

Suppose that we toss a coin 3 times, let *X* be the number of heads in 3 tosses. The table is:

| $X$ | $P(x)$ | $X*P(x)$ |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 3/8 | 3/8 |
| 2 | 3/8 | 6/8 |
| 3 | 1/8 | 3/8 |

$\mu_X$ =12/8=1.5. So the expected number of heads in three is one and a half heads.

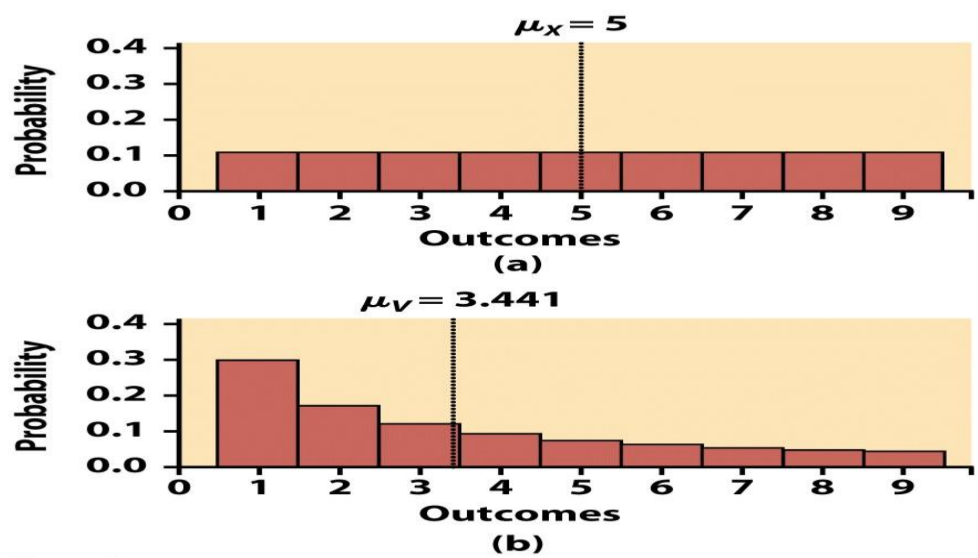


Figure 4-13
*Introduction to the Practice of Statistics, Fifth Edition*


*Figure* Locating the mean of a discrete random variable on the probability histogram for (a) digits between 1 and 9 chosen at random; (b) digits between 1 and 9 chosen from that obey Benford’s law.

If $p(x)$ is the pmf of $X$ and $h(X)$ is a function of $X$, then $E[h(X)]=\sum_{i=1}^{k} h(x_i)p(x_i)$.

**VARIANCE OF A DISCRETE RANDOM VARIABLE**

Suppose that $X$ is a discrete random variable whose distribution is

| Value of $X$ | $x_1$ | $x_2$ | $x_3$ | $\cdots$ | $x_k$ |
|---|---|---|---|---|---|
| Probability | $p_1$ | $p_2$ | $p_3$ | $\cdots$ | $p_k$ |

and that $\mu_X$ is the mean of $X$. The **variance** of $X$ is

$$\begin{aligned}\sigma_X^2 &= (x_1-\mu_X)^2p_1+(x_2-\mu_X)^2p_2+\cdots+(x_k-\mu_X)^2p_k \\ &= \sum(x_i-\mu_X)^2p_i\end{aligned}$$

The **standard deviation** $\sigma_X$ of $X$ is the square root of the variance.

**Definition, pg 300**
***Introduction to the Practice of Statistics, Fifth Edition***


- $\text{Var}(X) = E(X^2)-[E(X)]^2$

## ***Linear Functions of Random Variables***

### Rules of Expected Value

- For any constant $a$, $E(aX) = aE(X)$.
- For any constant $b$, $E(X+b) = E(X) + b$.
- So, $E(aX+b) = aE(X) + b$.

### Rules for Variance

- For any constants $a$ or $b$,

  $\text{Var}(aX + b) = a^2\text{Var}(X)$.

## Mean and Variance of a Bernoulli Variable

If $X$~Bernoulli($p$), then

- $EX = \mu_X = p$
- Var$X = \sigma^2_X = p(1-p)$.

## The Binomial Distribution

### Binomial Distributions

The distribution of the count $X$ of successes is called the **binomial distribution** with parameters $n$ and $p$. The parameter $n$ is the number of observations, and $p$ is the probability of a success on any single observation. The possible values of $X$ are the integers from 0 to $n$. $X$~B($n$, $p$).

### THE BINOMIAL SETTING

**1.** There are a fixed number $n$ of observations.

**2.** The $n$ observations are all independent.

**3.** Each observation falls into one of just two categories, which for convenience we call "success" and "failure."

**4.** The probability of a success, call it $p$, is the same for each observation.

**Definition, pg 335**
***Introduction to the Practice of Statistics, Fifth Edition***

Example (a) Toss a balanced coin 10 times and count the number *X* of heads. There are $n$=10 tosses. Successive tosses are independent. If the coin is balanced, the probability of a head is $p$=0.5 on each toss. The number of heads we observe has the binomial distribution B(10, 0.5).

## ***Finding binomial probabilities: Tables***

We find cumulative binomial probabilities for some values for $n$ and $p$ by looking up probabilities in **Appendix Table A.1 (pp.664)** in the back of the book. The entries in the table are the cumulative probabilities $P(X \leq x)$ of individual outcomes for a binomial random variable *X*.

Example A quality engineer selects an SRS of 10 switches from a large shipment for detailed inspection. Unknown to the engineer, 10% of the switches in the shipment fail to meet the specifications. What is the probability that no more than 1 of the 10 switches in the sample fails inspection?

(Solution) Let $X$ be the count of bad switches in the sample. The probability that the switches in the shipment fail to meet the specification is $p$=0.1 and sample size is $n$=10. Thus $X$~B($n$=10, $p$=0.1). We want to calculate

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

Let's look at page 664 in the **Table A.1** for this calculation, look $n$=10 and x=1 under $p$=0.10. Then we find

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= .736.$$

About 74% of all samples will contain no more than 1 bad switch.

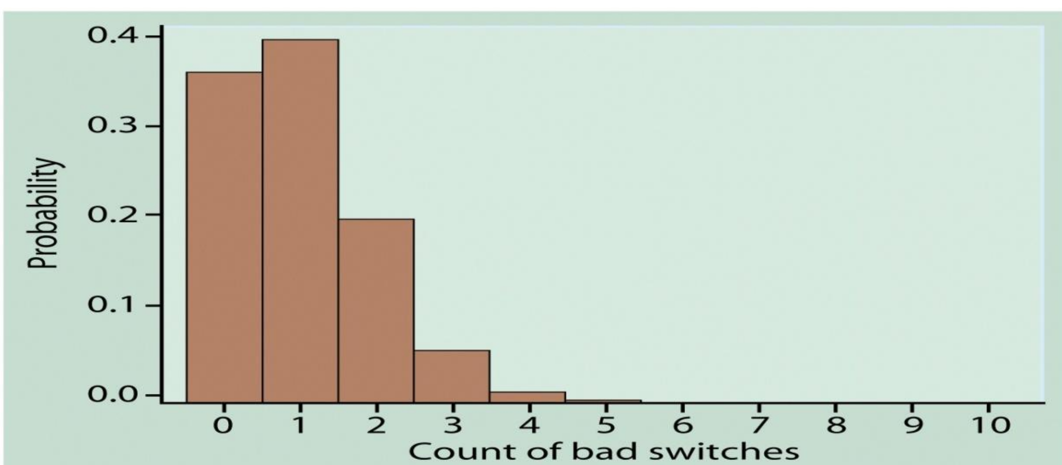


*Figure* Probability histogram for the binomial distribution with $n$=10 and $p$=0.1.

Example Corinne is a basketball player who makes 75% of her free throws over the course of a season. In a key game, Corinne shoots 15 free throws and misses 6 of them. The fans think that she failed because she was nervous. Is it unusual for Corinne to perform this poorly?

(Solution) Let $X$ be the number of misses in 15 attempts. The probability of a miss is p=1-0.75=0.25. Thus, $X$~B($n$=15, $p$=0.25).

We want the probability of missing 5 or more. Let's look at page 664 in the **Table A.1** for this calculation, look $n$=15 and $x$=5 under $p$=0.25.

$$P(X \geq 6) = P(X = 6) + \cdots + P(X = 15)$$

$$=1-P(X<6)=1-P(X\leq 5)=1-.852=.148.$$

Corinne will miss 6 or more out of 15 free throws about 15% of the time, or roughly one of every seven games. While below her average level, this performance is well within the range of the usual chance variation in her shooting.

## Binomial Mean and Standard Deviation

If a count X has the binomial distribution B($n$,$p$), then

$$\mu_X = n \times p$$

$$\sigma_X = \sqrt{n \times p \times (1 - p)}$$

## Binomial formulas

### BINOMIAL COEFFICIENT

The number of ways of arranging $k$ successes among $n$ observations is given by the **binomial coefficient**

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

for $k = 0, 1, 2, \ldots, n$.

**Definition, pg 349a**
***Introduction to the Practice of Statistics, Fifth Edition***