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Probability

In-class exercise

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The idea of this exercise is to gain familiarity with the distributions introduced in chapter four of Devore by collecting the various properties scattered through the text.

1. Fill in the table on page 2; (almost) all of the information should be available in sections 4.3-4.5 of Devore. Some notes

   - "IPs" and "XPs" refer to intrinsic and extrinsic parameters, respectively. Extrinsic parameters are those which just determine the origin and scale of the x axis, while intrinsic parameters control the shape of the pdf in ways that can't be transformed away by scaling and shifting x.
   - $\mathcal{N}$ is a normalization constant which doesn't contain x. The choice of this is not unique, but it should leave $f(x)/\mathcal{N}$ as simple as possible.
   - $f(x)/\mathcal{N}$ is what's left of the pdf after you factor out the constant.
   - In some cases the cdf $F(x)$ is not defined in closed form, but there is a special function defined for it. The special functions Devore uses are

     $$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du$$

     (which is related to the error function) and

     $$F(y;\alpha) = \int_{0}^{y} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}\, du$$

     (which is related the incomplete gamma function). In the case of the beta distribution, you can do the same thing and define something called the incomplete beta function, but Devore doesn't bother with this, and doesn't give the cdf for the beta distribution, so we'll just leave that entry blank (that's why it says **SKIP**).

   (a) The cdf, mean and variance for the chi-squared distribution are not listed in Devore, but we can still sort out what they are and write them down. That's because the pdf of the chi-squared distribution is actually a special form of another distribution for certain values of its parameters. From your table, determine which one that is and what the parameter values are, and use this information to complete the chi-squared row of the table.

2. Fill in the table on page 3, which collects the standard forms of the various distributions. You can do this from the table on page 2, using the specified values for the extrinsic parameters, and renaming x to y.

3. Each of the "standard" distributions is related to the corresponding general distribution. It defines the distribution of a random variable Y which is a linear transformation of the random variable X. For instance, for the normal distribution, $Y = \frac{X-\mu}{\sigma}$

Distribution	IPs	XPs	$\mathcal{N}$	$f(x)/\mathcal{N}$	$F(x)$	$x_{\min}$	$x_{\max}$	$E(X)$	$V(X)$
Normal	none	μ, $\sigma > 0$	$\frac{1}{\sigma\sqrt{2\pi}}$	$e^{-(x-\mu)^2/(2\sigma^2)}$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$	$-\infty$	∞	μ	σ^2
Exponential	none	λ	λ	$e^{-\lambda x}$	$1 - e^{-\lambda x}$	0	∞	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma	$\alpha > 0$	$\beta > 0$	$\frac{1}{\beta\Gamma(\alpha)}$	$\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta}$	$F\left(\frac{x}{\beta}; \alpha\right)$	0	∞	$\alpha\beta$	$\alpha\beta^2$
Weibull	$\alpha > 0$	$\beta > 0$	$\frac{\alpha}{\beta}$	$\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha}$	$1 - e^{-(x/\beta)^\alpha}$	0	∞	$\beta\Gamma\left(1 + \frac{1}{\alpha}\right)$	$\beta^2 \left\{\Gamma\left(1 + \frac{2}{\alpha}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha}\right)\right]^2\right\}$
Lognormal	$\sigma > 0$	μ	$\frac{1}{\sigma\sqrt{2\pi}}$	$\frac{1}{x} e^{-(\ln x - \mu)^2/(2\sigma^2)}$	$\Phi\left(\frac{\ln x - \mu}{\sigma}\right)$	0	∞	$e^{\mu + \sigma^2/2}$	$e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$
Beta	$\alpha > 0$ $\beta > 0$	A $B > A$	$\frac{1}{B-A} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$	$\left(\frac{x-A}{B-A}\right)^{\alpha-1} \left(\frac{B-x}{B-A}\right)^{\beta-1}$	**SKIP**	A	B	$A + \frac{(B-A)\alpha}{\alpha+\beta}$	$\frac{(B-A)^2\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Chi-Squared	$\nu \in \mathbb{Z}^+$	none	$\frac{1}{2\Gamma(\nu/2)}$	$\left(\frac{x}{2}\right)^{\nu/2-1} e^{-x/2}$	$F\left(\frac{x}{2}, \frac{\nu}{2}\right)$	0	∞	ν	2ν

Distribution	IPs	XPs	$\mathcal{N}$	$f(y)/\mathcal{N}$	$F(y)$	$y_{\min}$	$y_{\max}$	$E(Y)$	$V(Y)$
Standard Normal	none	$\mu = 0$ $\sigma = 1$	$\frac{1}{\sqrt{2\pi}}$	$e^{-y^2/2}$	$\Phi(y)$	$-\infty$	∞	0	1
Standard Exponential	none	$\lambda = 1$	1	e^{-y}	$1 - e^{-y}$	0	∞	1	1
Standard Gamma	$\alpha > 0$	$\beta = 1$	$\frac{1}{\Gamma(\alpha)}$	$y^{\alpha-1}e^{-y}$	$F(y;\alpha)$	0	∞	α	α
Standard Weibull	$\alpha > 0$	$\beta = 1$	α	$y^{\alpha-1}e^{-y^\alpha}$	$1 - e^{-y^\alpha}$	0	∞	$\Gamma\left(1 + \frac{1}{\alpha}\right)$	$\left\{\Gamma\left(1 + \frac{2}{\alpha}\right) - \left[\Gamma\left(1 + \frac{1}{\alpha}\right)\right]^2\right\}$
Standard Lognormal	$\sigma > 0$	$\mu = 0$	$\frac{1}{\sigma\sqrt{2\pi}}$	$\frac{1}{y}e^{-(\ln y)^2/(2\sigma^2)}$	$\Phi\left(\frac{\ln y}{\sigma}\right)$	0	∞	$e^{\sigma^2/2}$	$e^{\sigma^2}(e^{\sigma^2} - 1)$
Standard Beta	$\alpha > 0$ $\beta > 0$	$A = 0$ $B = 1$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$	$y^{\alpha-1}(1-y)^{\beta-1}$	**SKIP**	0	1	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Answers to the other questions:

1.(a) Inspection of the form of $f(x)$ shows that the chi-squared distribution is actually a gamma distribution with $\alpha = \nu/2$ and $\beta = 2$.

3. Comparing the cdfs (or pdfs) shows:

Normal:

$$Y = \frac{X - \mu}{\sigma}$$

(we usually call this Z rather than Y)

Exponential:

$$Y = \lambda X$$

Gamma and Weibull:

$$Y = \frac{1}{\beta} X$$

Lognormal:

$$\ln Y = \ln X - \mu$$

i.e.,

$$Y = Xe^{-\mu}$$

Beta:

$$Y = \frac{X - A}{B - A}$$