1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 1

Assigned 2010 September 7
Due 2010 September 14

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Separation of Variables

Show that the differential equation

$$\nabla^2\psi(r,\theta,\phi) + \left(k^2 + f(r) + \frac{1}{r^2}g(\theta) + \frac{1}{r^2\sin^2\theta}h(\phi)\right)\psi(r,\theta,\phi) = 0 \tag{1.1}$$

has solutions of the form

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \tag{1.2}$$

and work out the ordinary differential equations satisfied by $R(r)$, $\Theta(\theta)$ and $\Phi(\phi)$.

2 Solutions to the Boundary Value Problem

In class we showed that the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{2.1}$$

with the boundary condition $\psi(a,\phi,t) = 0$ and initial conditions $\psi(r,\phi,0) = f(r,\phi)$ and $\dot{\psi}(r,\phi,0) = 0$ had a solution of the form

$$\psi(r,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\gamma_{mn}}{a}r\right)(A_{mn}\cos m\phi + B_{mn}\sin m\phi)\cos\frac{\gamma_{mn}}{a}ct \tag{2.2}$$

where γ_{mn} is the nth non-trivial zero of the mth bessel function so $J_m(\gamma_{mn}) = 0$.

a) Use your favorite plotting program to plot $J_m\left(\frac{\gamma_{mn}}{a}r\right)$ versus r/a for various choices of m and n as follows:

   i) Plot $J_0\left(\frac{\gamma_{0n}}{a}r\right)$ for $n = 1, 2, 3$ on one set of axes.

   ii) Plot $J_1\left(\frac{\gamma_{1n}}{a}r\right)$ for $n = 1, 2, 3$ on one set of axes.

   iii) Plot $J_2\left(\frac{\gamma_{2n}}{a}r\right)$ for $n = 1, 2, 3$ on one set of axes.

   Please also hand in a printout of the commands used to do the plotting.

   In maplotlib, it's convenient to import from `scipy.special` the functions `jn` and `jn_zeros`. Note that e.g. `jn_zeros(2,3)` returns an array containing γ_{21}, γ_{22} and γ_{23}.

b) It's a property of Bessel functions that $J_m(0) = 0$ for $m > 0$; explain why that is necessary to ensure $\psi(r, \phi, t)$ is well-defined.

c) Suppose now that the boundary condition is $\frac{\partial\psi}{\partial r}\Big|_{r=a} \equiv \psi_{,r}(a, \phi, t) = 0$. Work out the solution to the wave equation in a form similar to (2.2) using the definition that ν_{mn} is the nth zero of the derivative $J'_m(x)$ of the mth Bessel function.

3 Orthogonality of Spherical Harmonics

Spherical harmonics

$$Y_\ell^m(\theta, \phi) = P_\ell^{|m|}(\cos\theta)e^{im\phi} \qquad m \in \{-\ell, -\ell+1, \dots \ell-1, \ell\}, \quad \ell = \{0, 1, 2 \dots\} \tag{3.1}$$

are eigenfunctions of the differential operators $\widehat{L^2}$ and $\widehat{L_z}$ defined by

$$\widehat{L^2}\psi = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \qquad \text{and} \qquad \widehat{L_z}\psi = -i\frac{\partial\psi}{\partial\phi} \tag{3.2}$$

with eigenvalues $\ell(\ell+1)$ and m, respectively:

$$\widehat{L^2}\, Y_\ell^m(\theta, \phi) = \ell(\ell+1)Y_\ell^m(\theta, \phi) \qquad \widehat{L_z}\, Y_\ell^m(\theta, \phi) = mY_\ell^m(\theta, \phi) \tag{3.3}$$

a) Show that $\widehat{L_z}$ is a Hermitian operator under the inner product

$$\langle u, v\rangle = \int_0^{2\pi}\int_0^{\pi} [u(\theta, \phi)]^*\, v(\theta, \phi)\, \sin\theta\, d\theta\, d\phi \tag{3.4}$$

   for functions $u(\theta, \phi)$ and $v(\theta, \phi)$ which are well-behaved on the sphere, i.e., that

$$\int_0^{2\pi}\int_0^{\pi} [u(\theta, \phi)]^* \left[\widehat{L_z}\, v(\theta, \phi)\right] \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} \left[\widehat{L_z}\, u(\theta, \phi)\right]^* v(\theta, \phi)\, \sin\theta\, d\theta\, d\phi \tag{3.5}$$

b) Show that $\widehat{L^2}$ is Hermitian under (3.4).

c) **EXPLAIN** why this implies an orthogonality relation for the $\{Y_\ell^m\}$, and write that relation.