1060-710
Mathematical and Statistical Methods for Astrophysics

Problem Set 2

Assigned 2010 September 14
Due 2010 September 21

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

0 Conventions

The convention we're using for the continuous Fourier transform is

$$\mathcal{F}\left\{h\right\} = \widetilde{h}(f) = \int_{-\infty}^{\infty} h(t)\, e^{-i2\pi f t}\, dt \tag{0.1a}$$

$$\mathcal{F}^{-1}\left\{\widetilde{h}\right\} = h(t) = \int_{-\infty}^{\infty} \widetilde{h}(f)\, e^{i2\pi f t}\, df \tag{0.1b}$$

and for the discrete Fourier transform

$$\widehat{h}_k = \sum_{j=0}^{N-1} h_j\, e^{-i2\pi jk/N} \tag{0.2a}$$

$$h_j = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{h}_k\, e^{i2\pi jk/N} \tag{0.2b}$$

1 The Forced Damped Harmonic Oscillator

A damped, driven harmonic oscillator is described by the differential equation

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + (2\pi f_0)^2 x(t) = F(t)/m \ . \tag{1.1}$$

a) Starting from the inverse Fourier transform (0.1b), determine expressions for $\dot{x}(t)$ and $\ddot{x}(t)$ in terms of the Fourier transform $\widetilde{x}(f)$.

b) Substitute your expressions from part a), along with the Fourier expansions of $x(t)$ and $F(t)$, into (1.1) to obtain an equation relating the Fourier transforms $\widetilde{x}(f)$ and $\widetilde{F}(f)$.

c) Solve this equation for $\widetilde{x}(f)$.

2 Convolution and Impulse Response

A quiescent damped harmonic oscillator, when struck at a time t' with an impulsive force $p_0\delta(t-t')$, will respond with a displacement $x(t) = p_0\, R(t-t')$ where

$$R(t-t') = \begin{cases} 0 & t < t' \\ \frac{1}{2\pi f_1 m} e^{-\gamma(t-t')} \sin 2\pi f_1(t-t') & t > t' \end{cases} \tag{2.1}$$

and $f_1 = \sqrt{f_0^2 - (\gamma/2\pi)^2}$. This means that, by the principle of superposition, the response to a general force

$$F(t) = \int_{-\infty}^{\infty} \delta(t-t')\, F(t')\, dt' \tag{2.2}$$

will be

$$x(t) = \int_{-\infty}^{\infty} R(t-t')\, F(t')\, dt' \tag{2.3}$$

a) Find the Fourier transform

$$\widetilde{R}(f) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i2\pi f\tau}\, d\tau \tag{2.4}$$

[Hint: the integral is easiest if you break up $R(\tau)$ using the fact that $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$; just be sure, at the end, to combine the two pieces and simplify as much as possible.]

b) Use the convolution theorem (which you don't need to prove again) to find an expression for $\widetilde{x}(f)$ in terms of $\widetilde{F}(f)$, using the the explicit form of $\widetilde{R}(f)$ from part a).

c) Compare the result of this problem to the result of problem 1

d) Extra credit: Writing $\widetilde{R}(f)$ as a real amplitude times a phase,

$$\widetilde{R}(f) = A(f) e^{i\phi(f)} \tag{2.5}$$

work out the real functions $A(f)$ and $\phi(f)$. [Extra extra credit: plot $(2\pi f_0)^2 m A(f)$ and $\phi(f)$ versus f/f_0 for $\gamma = 2\pi f_0/10$ and $\gamma = 2\pi f_0/20$.]

3 Parseval's Theorem

a) Use

$$\int_{-\infty}^{\infty} e^{-i2\pi(f-f')t}\,dt = \delta(f-f') = \delta(f'-f) \tag{3.1}$$

to show that

$$\int_{-\infty}^{\infty} g^*(t)\,h(t)\,dt = \int_{-\infty}^{\infty} \widetilde{g}^*(f)\,\widetilde{h}(f)\,df \tag{3.2}$$

b) Use

$$\sum_{k=0}^{N-1} e^{i2\pi(j-\ell)k/N} = N\,\delta_{j,\ell \bmod N} \tag{3.3}$$

to show that

$$\sum_{j=0}^{N-1} g_j^*\,h_j = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{g}_k^*\,\widehat{h}_k \tag{3.4}$$

4 Sampling and Aliasing

a) Consider the 1.5 Hz sine wave $h(t) = \sin(2\pi[1.5\,\mathrm{Hz}]t)$, sampled at $\delta t = 0.5\,\mathrm{s}$. Sketch the continuous function $h(t)$ from $t = 0$ to $t = 4.0\,\mathrm{s}$, and put dots at the samples $h_j = h(j\,\delta t)$.

b) $\{h_j\}$ is also the discretization of a lower-frequency sinusoidal function, i.e., $h_j = \mathfrak{h}(j\delta t)$. What is $\mathfrak{h}(t)$?

c) What are the Fourier components $\{\widehat{h}_k | k = 0, \ldots, 7\}$? What about $\{\widehat{h}_k | k = -4, \ldots, 3\}$? (Note: rather than calculating the forward Fourier transform (0.2a) analytically or numerically for each value of k, it's much easier to write h_j as a sum of exponentials and then read off the Fourier components using (0.2b).)