Conformal Mappings
(Zill & Wright Chapter 20)

1016-420-02: Complex Variables[*]

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Contents

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1 Motivation

We turn now from contour integration back to some ideas we first touched on when we introduced complex functions, specifically:

- A complex function $w = f(z)$ can be thought of as a mapping from the $z = x + iy$ plane with coördinates (x, y) to the $w = u + iv$ plane with coördinates (u, v).
- If $\varphi(x, y)+i\psi(x, y)$ is analytic, then the real functions $\varphi(x, y)$ and $\psi(x, y)$ are harmonic, i.e., each obeys the Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{1.1}$$

The main application of this will be in converting the Laplace equation with boundary conditions on an awkward region in the (x, y) plane to one on a more convenient region in the (u, v) plane. I.e., if we have a harmonic function $U(u, v)$ which is part of an analytic function

$$W = F(w) = U(u, v) + iV(u, v) \tag{1.2}$$

on some convenient region in the w plane and $w = f(z) = u(x, y) + iv(x, y)$ is an analytic function which maps the region of interest in the (x, y) plane onto the convenient region in the (u, v) plane, then the chain rule tells us that $F(f(z))$ is an analytic function of z; that means if we write

$$\varphi(x, y) + i\psi(x, y) := F(f(z)) = U(u(x, y), v(x, y)) + iV(u(x, y), v(x, y)) \tag{1.3}$$

then we can see that

$$\varphi(x, y) = U(u(x, y), v(x, y)) \tag{1.4}$$

is a harmonic function of x and y with boundary conditions on the appropriate region.

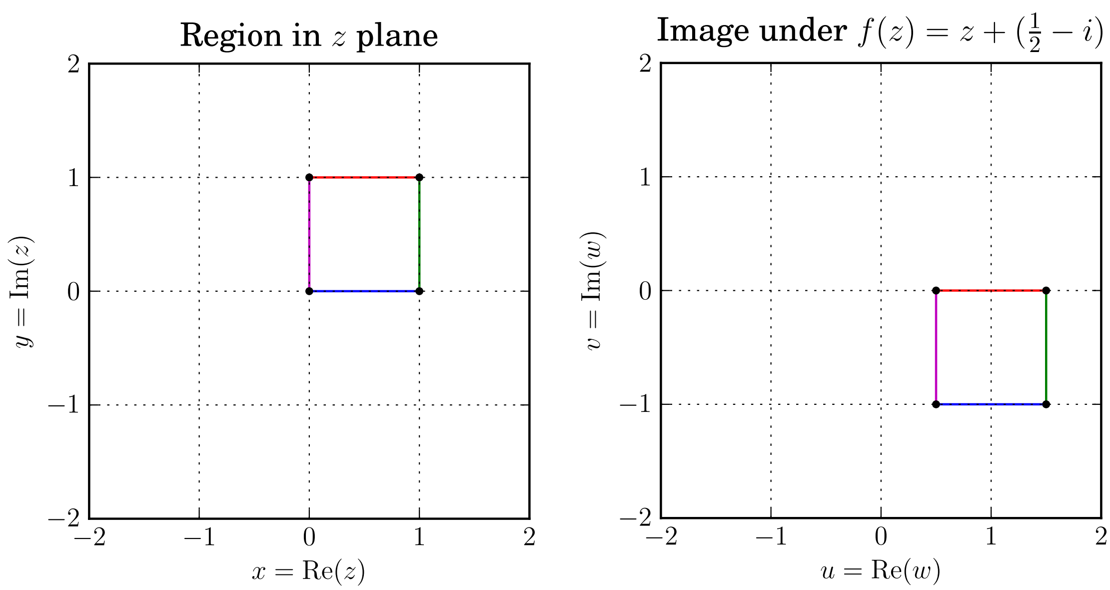
2 Complex Functions as Mappings

Recall that when we first introduced complex functions, we proposed the idea of mapping the z plane, with coördinates $x = \text{Re}(z)$ and $y = \text{Im}(z)$, into the w plane, with coördinates $u = \text{Re}(w)$ and $v = \text{Im}(w)$.

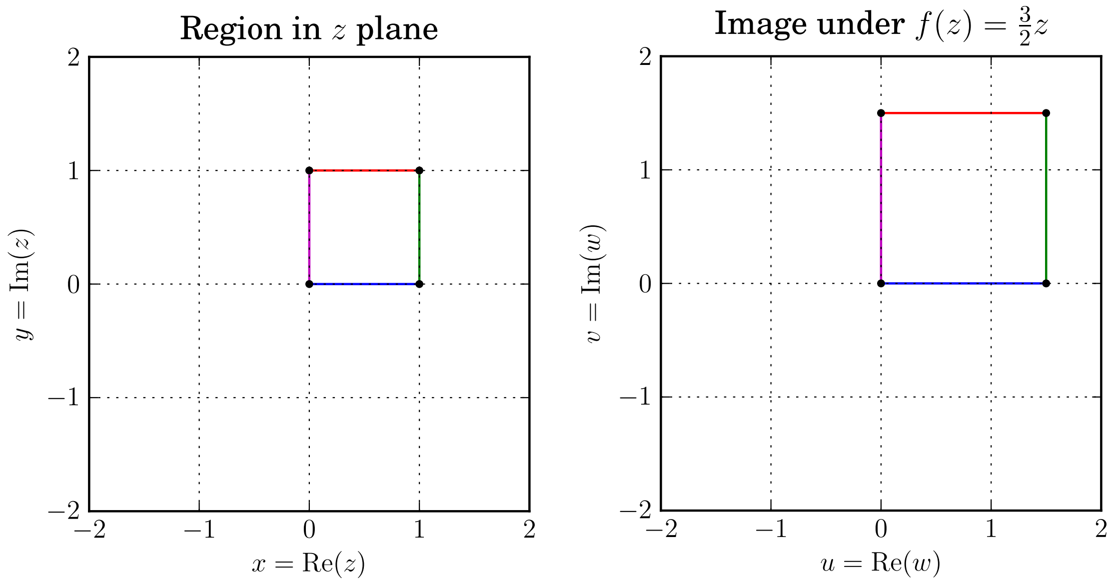
We now want to think about how this mapping affects not just curves, but regions of the (x, y) plane.

Translation: $f(z) = z + b$ where $b = h + ik$; then

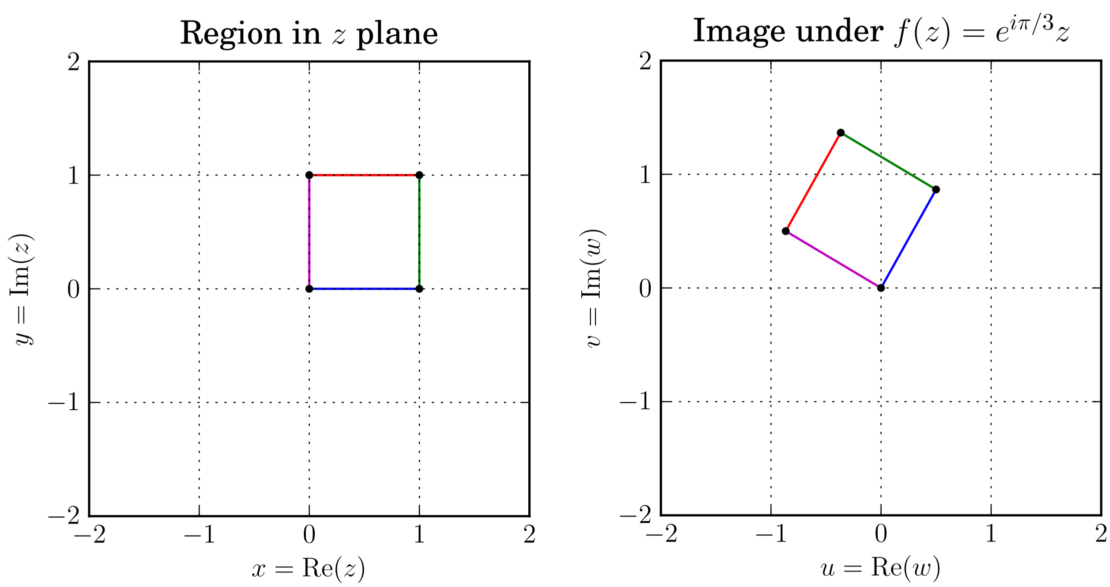
$$u = x + h \tag{2.1a}$$
$$v = y + k \tag{2.1b}$$



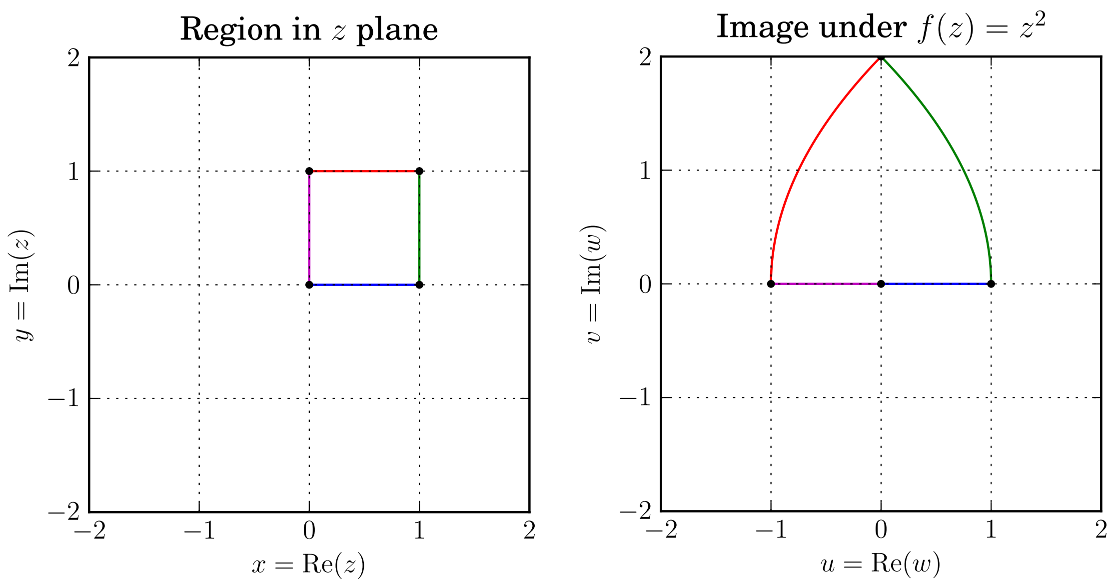
Magnification: $f(z) = \alpha z$ where α is a positive real constant.



Rotation: $f(z) = e^{i\theta_0} z$ where θ_0 is a real constant.



Effect of raising to a power: $f(z) = z^\alpha$ scales angles at $z = 0$ by a factor of α:



Practice Problems

20.1.1, 20.1.5, 20.1.7, 20.1.11, 20.1.13, 20.1.15, 20.1.21, 20.1.23, 20.1.25, 20.1.29

Thursday 3 November 2011

3 Conformal Mappings

3.1 Conditions for Conformality

Having considered complex functions as mappings from the (x, y) plane to the (u, v) plane, we now turn specifically to *conformal mappings*. "Conformal" means preserving angles, i.e., if two curves $z_1(t)$ and $z_2(t)$ intersect (at a point $z_0 = z_1(t_0) = z_2(t_0)$) at an angle α in the (x, y) plane, the mapped curves $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$ will intersect (at the point $f(z_0) = w_1(t_0) = w_2(t_0)$) at the same angle in the (u, v) plane. We can see what conditions on $f(z)$ achieve this by considering the heading of a curve $z(t)$, i.e., what angle its tangent vector makes to a line parallel to the real axis. Since the curve $z(t) = x(t) + iy(t)$ has x coördinate $x(t)$ and y coördinate $y(t)$, its tangent vector is

$$x'(t)\hat{x} + y'(t)\hat{y} \tag{3.1}$$

The heading of this vector is

$$\texttt{atan2}(y'(t), x'(t)) = \operatorname{Arg} z'(t) \tag{3.2}$$

so the angle at which the curves $z_1(t)$ and $z_2(t)$ intersect is the difference of their headings,

$$\alpha = \arg z_2'(t_0) - \arg z_1'(t_0) \tag{3.3}$$

where we have written arg rather than the principal value Arg because adding multiples of 2π to won't change its physical meaning.

Likewise, heading of the curve $w(t) = f(z(t))$ is $\operatorname{Arg} w'(t)$. The chain rule tells us that

$$w'(t) = f'(z(t))z'(t) \tag{3.4}$$

so the heading is

$$\arg w'(t) = \arg f'(z(t)) + \arg z'(t) \tag{3.5}$$

where we have used the fact that

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{3.6}$$

This means that the headings of the two curves $w_1(t)$ and $w_2(t)$ at the point of intersection are

$$\arg w_1'(t_0) = \arg f'(z_0) + \arg z_1'(t_0) \tag{3.7a}$$

$$\arg w_2'(t_0) = \arg f'(z_0) + \arg z_2'(t_0) \tag{3.7b}$$

which means the angle between them is

$$\arg w_2'(t_0) - \arg w_1'(t_0) = \arg z_2'(t_0) - \arg z_1'(t_0) \tag{3.8}$$

This demonstration works as long as $\arg f'(z_0)$ is well defined. So the function $f(z)$ has to be analytic (so that $f'(z)$ exists along the curves), but also $f'(z_0)$ must be non-zero, because $\arg 0$ is undefined. I.e.,

$$f(z) \text{ defines a conformal mapping wherever } f(z) \text{ is analytic and } f'(z) \neq 0 \tag{3.9}$$

Note that this was the case with our example of $f(z) = z^2$ on Tuesday. Since $f'(z) = 2z$ which is zero only at $z = 0$, the mapping is conformal except at $z = 0$. We saw this, as the right angles at the corners of the square remained right angles, except for the one at the origin.

3.2 Application to Dirichlet Problems

Example 20.2.19.

4 Geometric Applications of Analytic Functions

4.1 Potential for a Vector Field

Recall that when we first defined a complex function

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{4.1}$$

we deduced the Cauchy-Riemann equations from

$$f'(z) = \frac{\partial}{\partial x} f(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial}{\partial y} f(x+iy) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{4.2}$$

We also saw that the Cauchy-Riemann equations meant that the Pólya vector field

$$\vec{H} = u(x,y)\hat{x} - v(x,y)\hat{y} \tag{4.3}$$

had zero divergence and zero curl:

$$\operatorname{div}\vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{4.4a}$$

$$\operatorname{curl}\vec{H} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{4.4b}$$

Another fact from vector calculus is that any vector field with zero curl can be written as a gradient of some scalar field, i.e., there should be a $\varphi(x,y)$ such that

$$\vec{H} = \tilde{\nabla}\,\varphi = \frac{\partial \varphi}{\partial x}\hat{x} + \frac{\partial \varphi}{\partial y}\hat{y} \tag{4.5}$$

We can construct this field by starting with the analytic function $f(z)$. Since it is analytic, we can construct an antiderivative $F(z)$ such that $f(z) = F'(z)$. Write the real and imaginary parts of the antiderivative as

$$F(z) = F(x+iy) = \varphi(x,y) + i\psi(x,y) \tag{4.6}$$

We note that we can write the derivative as

$$F'(z) = \frac{\partial}{\partial x} F(x+iy) = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{1}{i}\frac{\partial}{\partial y} F(x+iy) = \frac{\partial \psi}{\partial y} - i\frac{\partial \varphi}{\partial y} \tag{4.7}$$

The Cauchy-Riemann equations applied to $F(z)$ are

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{4.8a}$$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \varphi}{\partial y} \tag{4.8b}$$

so we can also write

$$f(z) = F'(z) = \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y} \tag{4.9}$$

That makes the Pólya vector field for $f(z)$

$$\vec{H} = \frac{\partial \varphi}{\partial x}\hat{x} + \frac{\partial \varphi}{\partial y}\hat{y} = \tilde{\nabla}\,\varphi \tag{4.10}$$

The scalar field φ, which is harmonic, is called the potential for the Pólya vector field $\vec{H}$.

4.2 Sample Application

Example 20.6.5

Practice Problems

20.2.1, 20.2.3, 20.2.5, 20.2.7, 20.2.9, 20.2.19, 20.2.23, 20.6.1, 20.6.3, 20.6.5