Functions of a Complex Variable (Zill & Wright Chapter 17)

1016-420-02: Complex Variables*

Winter 2012-2013

Contents

Tuesday 27 November 2012

0 Administrata

- Syllabus
- The last letter of the English alphabet, z, can be pronounced "zee" or "zed", depending on your dialect.
- Instructor's name (Whelan) rhymes with "wailin'".
- Text: Zill and Wright, *Advanced Engineering Mathematics*. You'll need to sign up for WebAssign access, but note that this can be purchased separately from the book. You should have access to the book as an e-book via WebAssign. There is also a copy of the book on reserve at the Library.
- Course website: `http://ccrg.rit.edu/~whelan/1016-420/`
  - Contains links to exams from the Fall 2011 sections of this course.
- Course calendar: *tentative* timetable for course.
- Structure:
  - Read relevant sections of textbook before class
  - Lectures to reinforce and complement the textbook
  - Practice problems (odd numbers; answers in back but more useful if you try them before looking!).
  - In-class exercises and simple quizzes; intended either to check basic concepts from assigned reading, or to emphasize ideas through active learning.
  - Weekly homework via WebAssign.
    - Note: doing the problems is *very* important step in mastering the material.
    - Note that you get 100 tries for each non-multiple-choice question, so you should be able to get most or all correct with sufficient effort.
    - In addition, you can earn 10% extra credit on a problem set by turning in a complete, neatly handwritten solution to your WebAssign problems, due at the beginning of class on the due date. This is useful practice for the exams!
    - Most of the questions are based on the "Review Questions" at the end of each chapter. The odd-numbered review questions have answers in the back of the book, but note that the WebAssign versions have had some numbers changed.
    - Numbers in the questions are "randomized" so that each student will get a slightly different version of the same problem, and therefore a different right answer!
  - Prelim exam (think midterm, but there are two of them) in class at end of each half of course: closed book, one handwritten formula sheet, use scientific calculator (*not* your phone!)
  - Final exam to cover both halves of course

- Grading:

  5% In Class Exercises/Quizzes
  10% Problem Sets
  25% First Prelim Exam
  25% Second Prelim Exam
  35% Final Exam

  You'll get a separate grade on the "quality point" scale (e.g., 2.5–3.5 is the B range) for each of these five components; course grade is weighted average.

0.1 Outline

Part One:

- Functions of a Complex Variable (Chapter 17)
- Integration in the Complex Plane (Chapter 18)

Part Two:

- Series and Residues (Chapter 19)
- Conformal Mappings (Chapter 20)

1 Complex Numbers

Ordinary numbers found on the number line, like 1, 42, 0, -12, 1.25, π, $-\frac{1}{3}$, $\sqrt{2}$, are called real numbers. The set of all real numbers is called $\mathbb{R}$ and it includes as subsets rational numbers, irrational numbers, integers, positive numbers, negative numbers, etc. When working with real numbers, we can add, subtract, multiply, and divide them (except division by zero) and the result is another real number. But we cannot solve an equation like $x^2 + 1 = 0$, because there is no real number whose square is -1. So we invent a new number called i, defined by $i^2 = -1$.[1] We can then construct numbers of the form $z = x + iy$, where x and y are both real numbers.[2] (I.e., $x, y \in \mathbb{R}$.) Such numbers are called *complex numbers*, and the set of all complex numbers is called $\mathbb{C}$. This course will be concerned with complex functions of complex numbers, i.e. functions of the form $w = f(z)$, where $w, z \in \mathbb{C}$. The resulting field of *Complex Analysis* will allow us to solve problems involving real functions, and have numerous applications in science and engineering.

But first we will consider the algebra of complex numbers. It turns out that if you add, subtract, multiply and divide complex numbers (except division by $0 = 0 + 0i$), the result will always be a complex number; by applying $i^2 = -1$ you can always get it back into the form $x + iy$, where x and/or y may be negative or zero. We can write down "rules" for this, but in fact most of it follows

[1] In some engineering textbooks, this number is called j; other less common notations are I and even J. In python, it is written `1j`.

[2] Another possible notation for $z = x+iy$ is (x, y); we won't typically use this in algebraic formulas, but it's useful to keep in mind for e.g., representing complex numbers on a computer.

from the extension of standard algebra and application of $i^2 = -1$. So you can work out some of these operations in the following exercise:

In-class exercise

Here are the general formulas for the operations you've carried out:

- Addition:
$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \tag{1.1a}$$
- Subtraction:
$$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2) \tag{1.1b}$$
- Multiplication:
$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + i(x_1y_2 + y_1x_2) + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \tag{1.1c}$$
Note that $(x+iy)(x-iy) = x^2 - (iy)^2 = x^2 - (-1)y^2 = x^2 + y^2$ which is a positive real number (unless $x = 0 = y$, in which case it's zero).
- Division:
$$\begin{aligned}\frac{x_1 + iy_1}{x_2 + iy_2} &= \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)\left(\frac{x_2 - iy_2}{x_2 - iy_2}\right) = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2)^2 + (y_2)^2} \\ &= \frac{(x_1x_2 + y_1y_2) + i(-x_1y_2 + y_1x_2)}{(x_2)^2 + (y_2)^2} = \frac{x_1x_2 + y_1y_2}{(x_2)^2 + (y_2)^2} + i\frac{-x_1y_2 + y_1x_2}{(x_2)^2 + (y_2)^2}\end{aligned} \tag{1.1d}$$
Note that in practice no one memorizes (or even looks up) (1.1d); we just apply the trick of multiplying and dividing by $x_2 - iy_2$ whenever we have a fraction with $x_2 + iy_2$ in the denominator.

Another useful fact is that $(-i)^2 = (-1)^2i^2 = i^2 = -1$. This means that in some sense i and $-i$ are on the same footing as square roots of -1. Because of this fact, an interesting operation is the *complex conjugate*, which switches i and $-i$. The complex conjugate of a complex number z is written[3] $\bar{z}$ and is defined as
$$\bar{z} = \overline{x + iy} = x - iy \tag{1.2}$$

The two real numbers which are equivalent to a complex number $z = x + iy$ are the real part $\mathrm{Re}(z) = x$ and the imaginary part $\mathrm{Im}(z) = y$. Note that the imaginary part of z is a real number! We can also write these using the complex conjugate:

$$\frac{z + \bar{z}}{2} = \frac{(x + iy) + (x - iy)}{2} = x = \mathrm{Re}(z) \tag{1.3a}$$
$$\frac{z - \bar{z}}{2} = \frac{(x + iy) - (x - iy)}{2i} = y = \mathrm{Im}(z) \tag{1.3b}$$

[3]In many applications, typically outside of complex analysis courses, the complex conjugate is written z^* instead of $\bar{z}$.

Another handy fact is that $i(-i) = -i^2 = -(-1) = 1$. This is part of the reason that, as noted before,

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 \tag{1.4}$$

which is a non-negative real number. This is the square of what we call the *modulus*

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} \tag{1.5}$$

The modulus is the complex generalization of the absolute value. Note that if $y = 0$ (so that z is the real number x), the modulus is $\sqrt{x^2} = |x|$.

We can perform most of the same operations and manipulations on complex numbers that we can on real numbers. Two notable exceptions:

- There is no concept of one complex number being greater than or less than another. While equality and inequality are well defined:

$$x_1 + iy_1 = x_2 + iy_2 \qquad \text{iff} \qquad x_1 = x_2 \text{ and } y_1 = y_2 \tag{1.6a}$$

$$x_1 + iy_1 \neq x_2 + iy_2 \qquad \text{iff} \qquad x_1 \neq x_2 \text{ or } y_1 \neq y_2 \ , \tag{1.6b}$$

  the symbols $>$, $<$, $\geq$, and $\leq$ have no meaning when applied to complex numbers.

- We only know how to take the square root of a non-negative real number. The symbol $\sqrt{}$ will **never ever ever** be applied to anything but a non-negative real number. We will instead use fractional powers, so for example $w = z^{1/2}$ means that w is any number such that $w^2 = z$. For example, we've seen that $i^2 = -1 = (-i)^2$, so we know that

$$(-1)^{1/2} = i \text{ or } -i \tag{1.7}$$

  The expression $z^{1/2}$ is not single-valued. (In fact, we'll see that if $z \neq 0$, there are exactly two possible values for $z^{1/2}$.) This is our first example of a multi-valued function; if we want to be careful, we can think about an expression like $(-1)^{1/2}$ as referring not to a single number, but to the *set* of values whose square is -1:

$$(-1)^{1/2} \equiv \{i, -i\} \tag{1.8}$$

  Note that this also applies to positive numbers. For example, $2^2 = 4 = (-2)^2$ so

$$4^{1/2} \equiv \{2, -2\} \ . \tag{1.9}$$

  Similarly,

$$2^{1/2} \equiv \{\sqrt{2}, -\sqrt{2}\} \ . \tag{1.10}$$

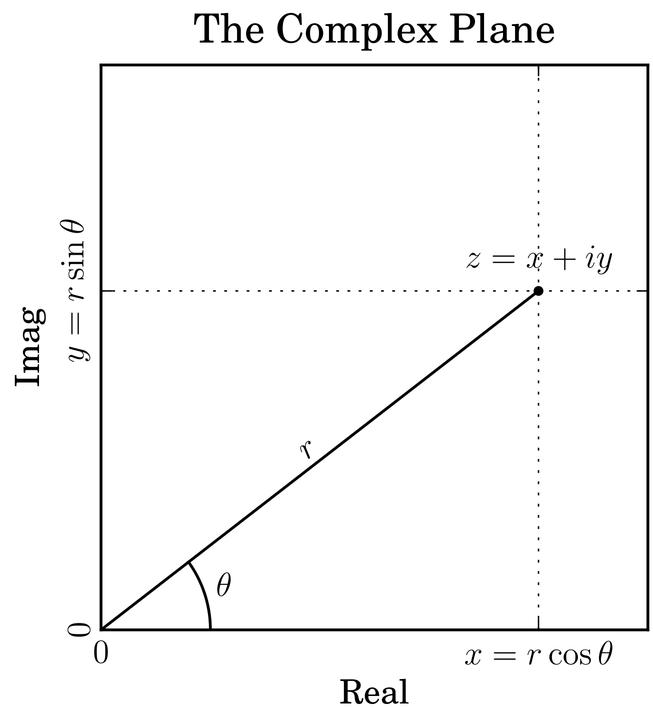
  Note the distinction. $\sqrt{2}$ is the single positive number 1.414..., while $2^{1/2}$ is multi-valued, and refers to the set consisting of $\sqrt{2}$ or $-\sqrt{2}$.

  Exercise: What is $i^{1/2}$? I.e., what are the possible combinations of x and y so that

$$(x + iy)^2 = (x + iy)(x + iy) = i \ ? \tag{1.11}$$

2 The Complex Plane

Recall that a complex number $z = x + iy$ can also be thought of as the ordered pair of real numbers (x, y). It is useful to associate the complex number $z = x + iy$ with the point with Cartesian coördinates (x, y). Each complex number can then be thought of as a point in the "complex plane" with the corresponding coördinates.



2.1 Polar Form of a Complex Number

An alternative to Cartesian coördinates (x, y) which is often useful is polar coördinates (r, θ), defined by

$$x = r\cos\theta \tag{2.1a}$$

$$y = r\sin\theta \tag{2.1b}$$

In terms of the polar coördinates (r, θ), the complex number is

$$z = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{2.2}$$

In the last step of (2.2) we have used the *Euler Relation*

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.3}$$

Note that this **only** works if we write θ in radians, *not* degrees! (The same is true for the usual expressions for derivatives of sine, cosine and other trig functions.) We will use radians exclusively in this course, and only refer to degrees momentarily to remind ourselves what an angle like $\pi/3$ or $\pi/6$ looks like in a triangle.

2.1.1 Aside: Complex Exponentials, Taylor Series, and the Euler Relation

We will now show that the Euler relation follows from the Taylor series for the sine, cosine and exponential functions. Recall the definition of the Taylor series[4]

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{2.4}$$

Apply this to three functions: $e^{i\theta}$, $\cos\theta$, and $\sin\theta$:

$f(\theta)$	$e^{i\theta}$	$\cos\theta$	$\sin\theta$
$f(0)$	1	1	0
$f'(\theta)$	$ie^{i\theta}$	$-\sin\theta$	$\cos\theta$
$f'(0)$	i	0	1
$f'''(\theta)$	$-e^{i\theta}$	$-\cos\theta$	$-\sin\theta$
$f'''(0)$	-1	-1	0
$f^{(4)}(\theta)$	$-ie^{i\theta}$	$\sin\theta$	$-\cos\theta$
$f^{(4)}(0)$	$-i$	0	-1
$f^{(n)}(0)(n \text{ even})$	$(-1)^{n/2} = i^n$	$(-1)^{n/2} = i^n$	0
$f^{(n)}(0)(n \text{ odd})$	$-1^{(n-1)/2}i = i^n$	0	$-1^{(n-1)/2} = i^{n-1}$

So the three Taylor series are

$$e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{3!}\theta^3 + \ldots \tag{2.5a}$$

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \ldots \tag{2.5b}$$

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \ldots \tag{2.5c}$$

from which, along with

$$i\sin\theta = i\theta - \frac{i}{3!}\theta^3 + \ldots \tag{2.6}$$

we see

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.7}$$

This is called the *Euler relation.*

Some useful examples:

$$e^{i0} = \cos 0 + i\sin 0 = 1 + i0 = 1 \tag{2.8a}$$

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i1 = i \tag{2.8b}$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0 = -1 \tag{2.8c}$$

$$e^{-i\pi/2} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i(-1) = -i \tag{2.8d}$$

$$2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2} \tag{2.8e}$$

[4] This Taylor expansion about $x = 0$ is more formally called the MacLaurin Series.

Practice Problems

17.1.1, 17.1.5, 17.1.9, 17.1.13, 17.1.15, 17.1.27, 17.1.35, 17.1.39

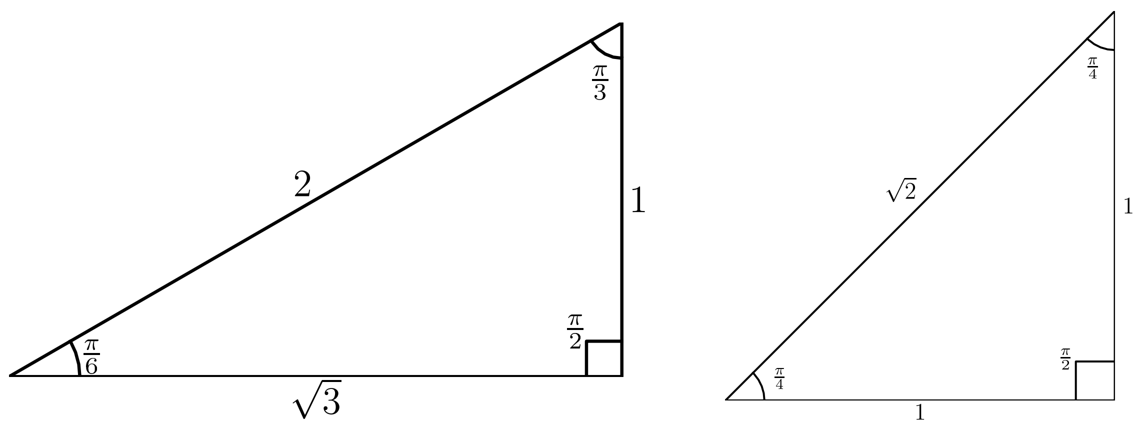
Thursday 29 November 2012

Handy table:

θ	$-5\pi/6$	$-3\pi/4$	$-2\pi/3$	$-\pi/2$	$-\pi/3$	$-\pi/4$	$-\pi/6$
$\theta\times(360°/2\pi)$	$-150°$	$-135°$	$-120°$	$-90°$	$-60°$	$-45°$	$-30°$
$\sin\theta$	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$
$\cos\theta$	$-\sqrt{3}/2$	$-\sqrt{2}/2$	$-1/2$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\theta\times(360°/2\pi)$	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Triangle mnemonics:

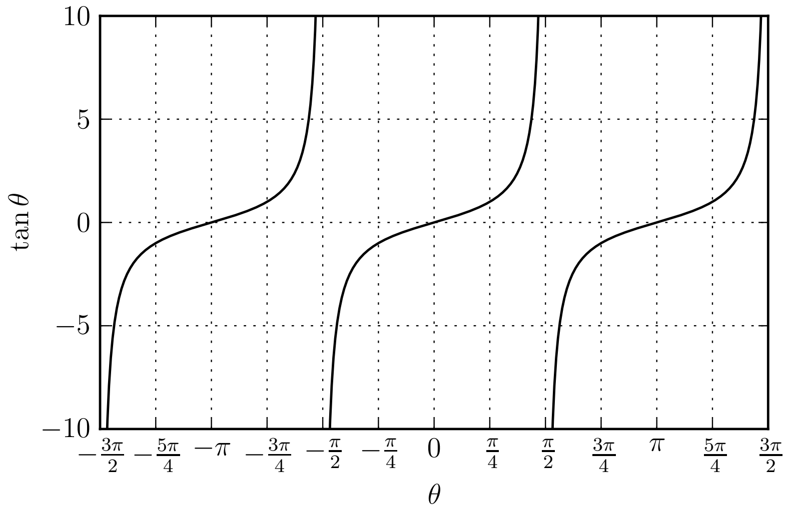


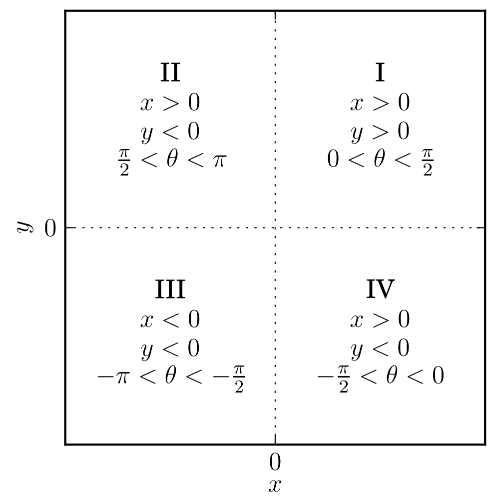
2.1.2 Converting Between Cartesian and Polar Representations

While the relations (2.1) tell us how to go from polar coördinates (r, θ) to Cartesian coördinates (x, y), or equivalently between the two representations of a complex number $x + iy = z = re^{i\theta}$, some care has to be taken in converting the other way. By the Pythagorean theorem, the radial coördinate r is the modulus of the complex number z

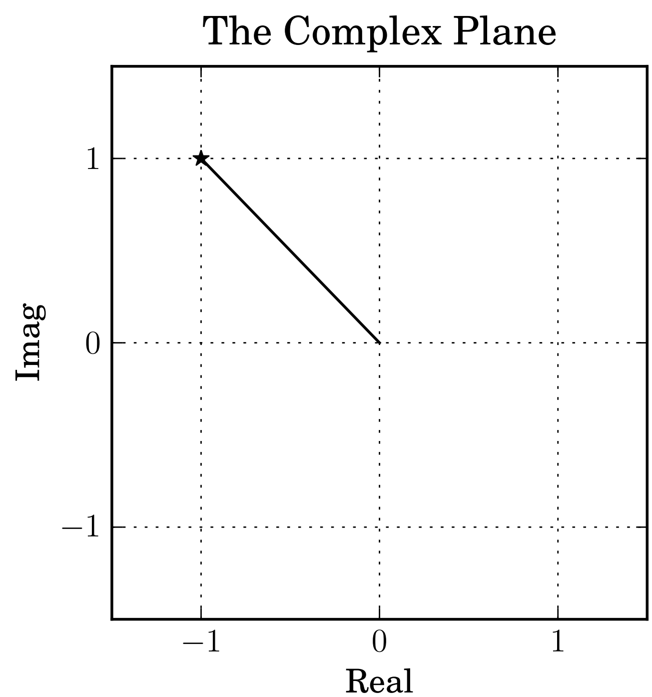
$$r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} = |z| \tag{2.9}$$

The angular coördinate θ is called the argument of z. Although we can solve (2.1) for $\tan\theta = y/x$, we can't simply identify θ with $\tan^{-1}(y/x)$. This is because we can achieve any value of $\tan\theta$ with a θ value between $-\pi/2$ and $\pi/2$:





Therefore simply asking for $\tan^{-1}(y/x)$ will never give us θ values corresponding to the second and third quadrants. For example, consider the case $x = -1$, $y = 1$. Looking at this point on a graph, it's pretty apparent that $r = \sqrt{2}$ and $\theta = 3\pi/4$:



However, if we try to calculate $\tan^{-1}(y/x)$ we get $\tan^{-1}(-1) = -\pi/4$, which would correspond to $x = 1$, $y = -1$. The solution to this is actually well-known in many computer languages; the function

$$\texttt{atan2}(y,x) = \begin{cases} \tan^{-1}(y/x) - \pi & x < 0 \text{ and } y < 0 \\ -\pi/2 & x = 0 \text{ and } y < 0 \\ \tan^{-1}(y/x) & x > 0 \\ \pi/2 & x = 0 \text{ and } y > 0 \\ \tan^{-1}(y/x) + \pi & x < 0 \text{ and } y \geq 0 \end{cases} \tag{2.10}$$

will always give the correct polar coördinate angle θ. (It is undefined when $x = 0 = y$.) The conversion from Cartesian to polar coördinates and back should thus be written and remembered as

$$x = r\cos\theta \tag{2.11a}$$
$$y = r\sin\theta \tag{2.11b}$$
$$r = \sqrt{x^2 + y^2} \tag{2.12a}$$
$$\theta = \texttt{atan2}(y,x) \tag{2.12b}$$

The following handout gives some tips on using geometry to convert between the Cartesian and polar representations of a complex number.

2.1.3 The Argument of a Complex Number

Note that, because $\cos\theta$ and $\sin\theta$ are periodic with period 2π, there are actually many choices of θ which give the same $z = re^{i\theta}$. For example, $e^{i3\pi/2} = -i = e^{-i\pi/2}$ and $e^{i2\pi} = 1 = e^{i0}$. We can define a multi-valued function known as the *argument*, written"arg":

$$\arg z \equiv \{ \quad \theta \quad | \quad z = |z|\, e^{i\theta} = |z|\,(\cos\theta + i\sin\theta) \quad \} \tag{2.13}$$

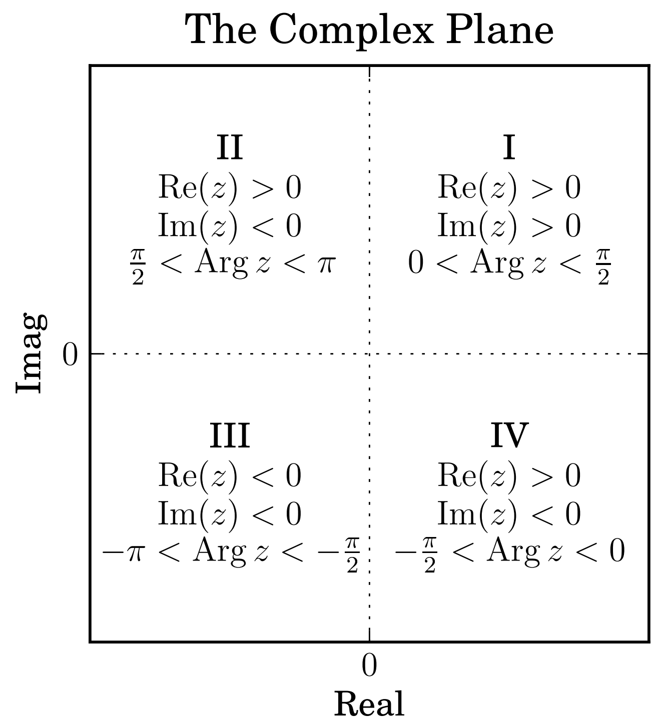
For example,

$$\arg i \equiv \left\{\ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots\right\} = \left\{\frac{\pi}{2} + k\,2\pi \,\middle|\, k \in \mathbb{Z}\right\} \tag{2.14}$$

For any z there is a choice of $\theta = \arg z$ such that $-\pi < \theta \leq \pi$. We call this the *principal value* of the argument and write it $\operatorname{Arg} z$; the multi-valued $\arg z$ represents this plus any integer multiple of 2π

$$\operatorname{Arg} z = \texttt{atan2}(y,x) \tag{2.15a}$$
$$\arg z \equiv \{\operatorname{Arg} z + k\,2\pi \mid k \in \mathbb{Z}\} \tag{2.15b}$$

The value of $\operatorname{Arg} z$ divides the complex plane into quadrants:



Note that $\operatorname{Arg} z$ is discontinuous on the negative real axis; $\operatorname{Arg}(-1) = \pi$, but $\operatorname{Arg}(-1+i\epsilon) \approx -\pi + \epsilon$ for small positive ϵ.

There is an important consequence of the periodicity of $re^{i\theta}$. We've already noted that, for real x_1, y_1, x_2, and y_2,

$$x_1 + iy_1 = x_2 + iy_2 \quad \text{if and only if} \quad x_1 = x_2 \text{ and } y_1 = y_2 \tag{2.16}$$

when comparing two complex numbers written in polar form, we have to be a bit careful:

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \quad \text{if and only if} \quad r_1 = r_2 \text{ and } \theta_1 = \theta_2 + k\,2\pi \text{ where } k \in \mathbb{Z} \tag{2.17}$$

Recall that the complex conjugate of $z = x + iy$ is $\bar{z} = x - iy$. If we write z in polar form

$$z = re^{i\theta} = (r\cos\theta) + i(r\sin\theta) \tag{2.18}$$

its complex conjugate is

$$\bar{z} = (r\cos\theta) + i(-r\sin\theta) = r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta} \tag{2.19}$$

I.e.,

$$|\bar{z}| = |z| \tag{2.20a}$$

$$\arg(\bar{z}) = -\arg(z) \tag{2.20b}$$

(When we say that a multi-valued expression like $\arg(\bar{z})$ equals another multi-valued expression like $-\arg(z)$, we mean that they each refer to the same set of values.)

Question: is $\operatorname{Arg}(\bar{z})$ always equal to $-\operatorname{Arg}(\bar{z})$?

2.2 Roots and Powers of a Complex Number

We've already looked at multiplication and division of complex numbers in Cartesian form. It turns out to be much simpler in polar form. Given two complex numbers

$$z_1 = r_1 e^{i\theta_1} \tag{2.21a}$$
$$z_2 = r_2 e^{i\theta_2} \tag{2.21b}$$

we have

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} \tag{2.22}$$

so

$$|z_1 z_2| = |z_1|\,|z_2| \tag{2.23a}$$
$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{2.23b}$$

Note that (2.23b) only holds in general for the multi-valued argument, not the principal value. For instance,

$$\operatorname{Arg}([-1][i]) = \operatorname{Arg}(-i) = -\frac{\pi}{2} \tag{2.24}$$

while

$$\operatorname{Arg}(-1) + \operatorname{Arg}(i) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}\ . \tag{2.25}$$

We can divide two complex numbers by noting

$$\frac{z_1}{z_2} = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right)^{-1} = \left(r_1 e^{i\theta_1}\right)\left(r_2^{-1} e^{-i\theta_2}\right) = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \tag{2.26}$$

so

$$|z_1/z_2| = |z_1|\,/\,|z_2| \tag{2.27a}$$
$$\arg(z_1/z_2) = \arg z_1 - \arg z_2\ . \tag{2.27b}$$

If we want to raise a complex number z to an integer power n, we again use the polar form:

$$z^n = (r\cos\theta + ir\sin\theta) = (re^{i\theta})^n = r^n e^{in\theta} = (r^n \cos n\theta + ir^n \sin n\theta) \tag{2.28}$$

so

$$|z^n| = |z|^n \tag{2.29a}$$
$$\arg(z^n) = n\arg z\ . \tag{2.29b}$$

In particular, we have

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = (\cos n\theta + i\sin n\theta)\ , \tag{2.30}$$

This is called *DeMoivre's Formula* and is proved with considerably more effort in Zill and Wright by using angle sum formulas.

The nth root of a complex number is written $z^{1/n}$, where n is an integer. It's defined, as for real numbers, to be the inverse of the nth power, i.e.,

$$w = z^{1/n} \text{ means } w^n = z \tag{2.31}$$

Recall that last time you were asked to find $z = x + iy = i^{1/2}$, defined by

$$i = (x + iy)^2 = (x^2 - y^2) + i(2xy) \tag{2.32}$$

This is equivalent to the system of two real equations

$$x^2 - y^2 = 0 \tag{2.33a}$$

$$2xy = 1 \tag{2.33b}$$

which has two solutions for (x, y): $(\sqrt{2}/2, \sqrt{2}/2)$ or $(-\sqrt{2}/2, -\sqrt{2}/2)$. This means

$$i^{1/2} \equiv \left\{ \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right\} \tag{2.34}$$

In practice, it's much easier to take roots of a complex number using the polar form, i.e., writing

$$z = re^{i\theta} = w^n = \rho e^{in\phi} \tag{2.35}$$

as equivalent to

$$w = \rho e^{i\phi} = z^{1/n} = (re^{i\theta})^{1/n} \tag{2.36}$$

It's tempting to just identify ρ with $\sqrt[n]{r}$ and ϕ with θ/n, but there's a catch. Because of the periodicity of the angle θ, the complex number z is unchanged if we add any multiple of 2π to θ. So

$$z = e^{i(\theta + k\,2\pi)} \tag{2.37}$$

where k is any integer. But

$$\frac{\theta + k\,2\pi}{n} = \frac{\theta}{n} + \frac{k}{n}\,2\pi \tag{2.38}$$

gives physically different angles if k/n is not an integer. This means that in general, if $z \neq 0$, there are n distinct roots $z^{1/n}$, given by

$$z^{1/n} = \left\{ \sqrt[n]{|z|}\; e^{i\left(\frac{\arg z}{n} + \frac{k}{n}\,2\pi\right)} \;\middle|\; k = 0, 1, \ldots, n-1 \right\} \tag{2.39a}$$

for $k = 0, 1, \ldots, n$.

Example: what is $(-8i)^{1/3}$? First we take the modulus and argument of $-8i$. Its modulus is 8 and its argument is $-\pi/2$. The three cube roots all have modulus $\sqrt[3]{8} = 2$ and their arguments are

$$\frac{1}{3}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{6} \tag{2.40a}$$

$$\frac{1}{3}\left(-\frac{\pi}{2} + 2\pi\right) = -\frac{\pi}{6} + \frac{2\pi}{3} = \frac{(4-1)\pi}{6} = \frac{\pi}{2} \tag{2.40b}$$

$$\frac{1}{3}\left(-\frac{\pi}{2} + 4\pi\right) = -\frac{\pi}{6} + \frac{4\pi}{3} = \frac{(8-1)\pi}{6} = \frac{7\pi}{6} = 2\pi - \frac{5\pi}{6} \tag{2.40c}$$

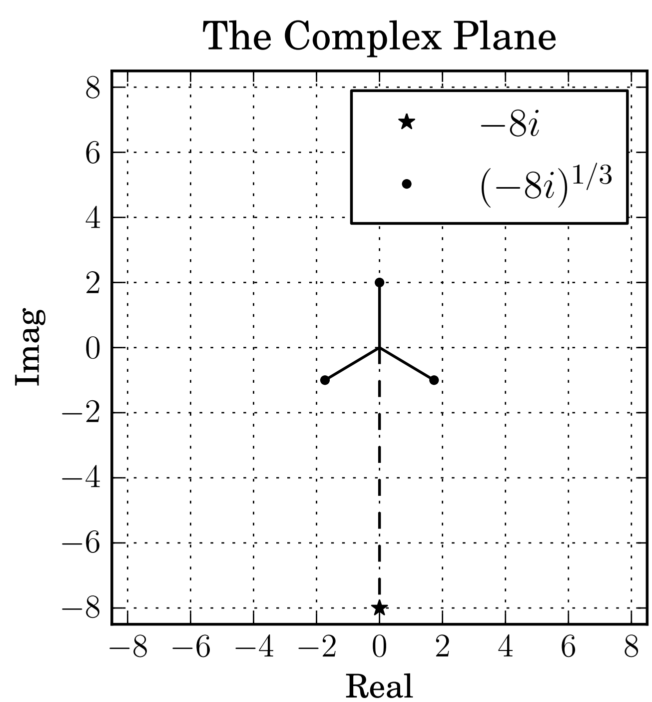
which makes the three cube roots (referring to the table at the beginning of today's notes)

$$2e^{-i\pi/6} = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right] = 2\left[\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] = \sqrt{3} - i \tag{2.41a}$$

$$2e^{i\pi/2} = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 2\left[0 + i\right] = 2i \tag{2.41b}$$

$$2e^{i7\pi/6} = 2e^{-i5\pi/6} = 2\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right] = 2\left[-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] = -\sqrt{3} - i \tag{2.41c}$$

If we look on the complex plane, we see that the roots are evenly spaced along a circle of radius 2:



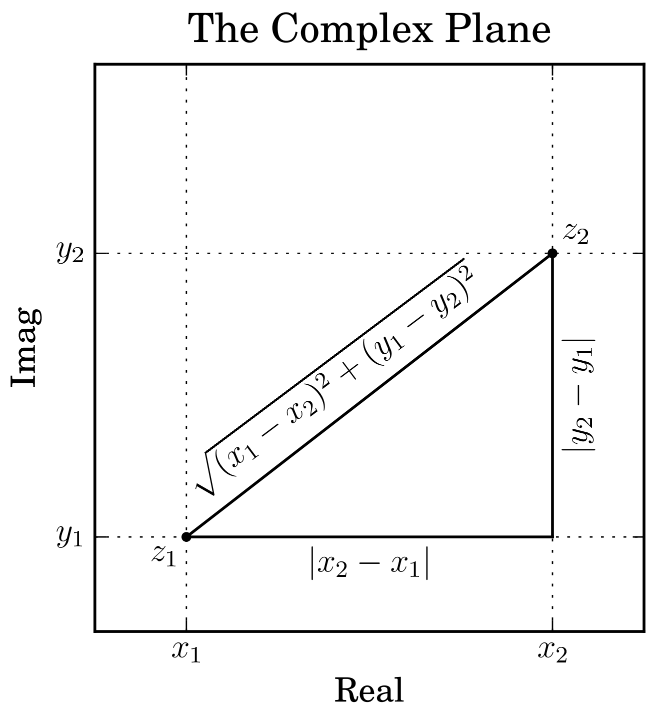
Practice Problems

17.2.7, 17.2.11, 17.2.13, 17.2.23, 17.2.33, 17.2.39, 17.3.5, 17.3.17

Tuesday 4 December 2012

2.3 Regions in the Complex Plane

Recall that we can't talk about one complex number being greater than or less than another. But we can apply these inequalities to real quantities constructed from complex numbers, like $|z|$, $\arg z$, $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$. This will be very useful, since it will allow us to talk about one complex number being "close to" another. If we think of complex numbers as points in the complex plane, the natural distance between two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is the distance between the points (x_1, y_1) and (x_2, y_2) arising from the Pythagorean theorem:



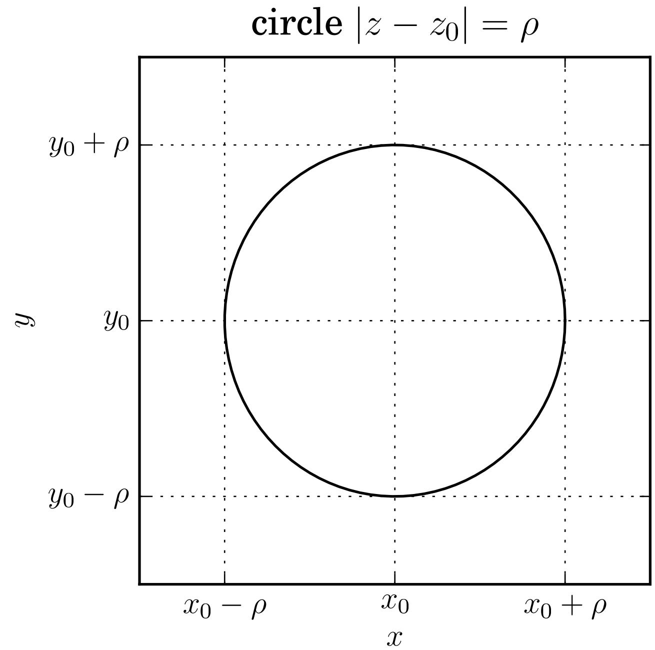
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |(x_2 - x_1) + i(y_2 - y_1)| = |z_2 - z_1| \tag{2.42}$$

If we take a fixed point z_0 in the complex plane, and a constant positive number ρ, then the equation

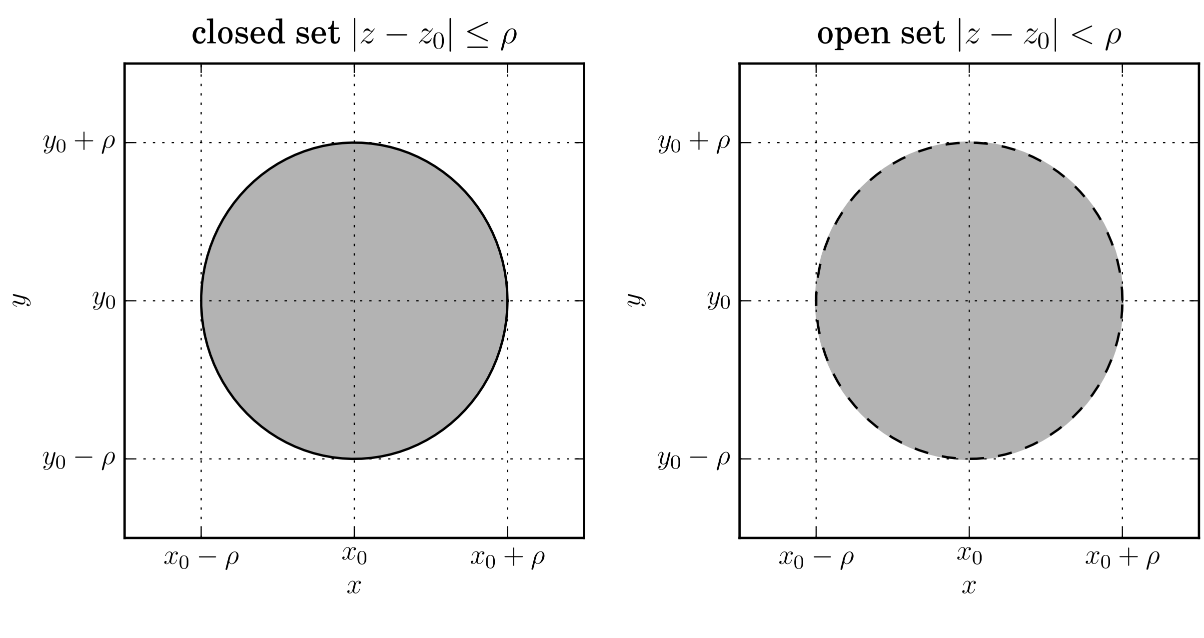
$$|z - z_0| = \rho \tag{2.43}$$

defines the set of all points z which are a distance ρ away from z_0. This is a circle of radius ρ centered on z_0, as we can easily see by writing $z = x + iy$ and $z_0 = x_0 + iy_0$ and squaring both sides of (2.43):

$$(x - x_0)^2 + (y - y_0)^2 = \rho^2 \tag{2.44}$$



If we replace the equation (2.43) with an inequality, we get a set of complex numbers, corresponding to a region in the complex plane. For example, $|z - z_0| \leq \rho$ is the interior of the circle of radius ρ centered at z_0, including its boundary. A set like this, which includes its boundary, is called a *closed set*. If we do not include the boundary $|z - z_0| = \rho$, we instead get what's called an *open set* $|z - z_0| < \rho$ which is the interior of the circle, not including its boundary:



We can create closed and open sets using a variety of inequalities. Each one is constructed from some real function(s) of the complex number z. See Zill and Wright for more examples.

3 Functions of a Complex Variable

3.1 Complex Functions

Recall the definition of a function in real analysis. A function f is a "machine" which takes a number x and spits out another number $f(x)$. So for example, if $f(x) = x^2 + 1$, then $f(2) = 5$. We talk about a function having a domain (the set of all numbers you can put into it) and a range (the set of all numbers that can come out of it). So for example if $f(x) = \sqrt{x}$ then the domain and range of f are both $[0, \infty)$.

In complex analysis, a function is a "machine" which takes a *complex* number z and spits out another complex number $w = f(z)$. So if $f(z) = z^2+1$ then $f(2) = 5$, $f(i) = 0$, and $f(1+i) = 1+2i$. Because output $w = u + iv$ of f has a real part and an imaginary part which both depend on the real and imaginary part of the input $z = x + iy$, we can also think of f as a machine which takes in two real numbers and spits out two real numbers:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) \tag{3.1}$$

For any complex function $f(z)$ there is a corresponding pair of real functions of two real variables, $u(x, y)$ and $v(x, y)$. For example, if $f(z) = z^2 + 1$,

$$u(x, y) + iv(x, y) = f(z) = (x + iy)^2 + 1 = x^2 + 2ixy - y^2 + 1 = (x^2 - y^2 + 1) + i(2xy) \tag{3.2}$$

which means that

$$u(x, y) = x^2 - y^2 + 1 \tag{3.3a}$$

$$v(x, y) = 2xy \tag{3.3b}$$

Similarly, given any two real functions $u(x, y)$ and $v(x, y)$, I can construct a complex function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, although the result might not be writable as a simple expression involving z.

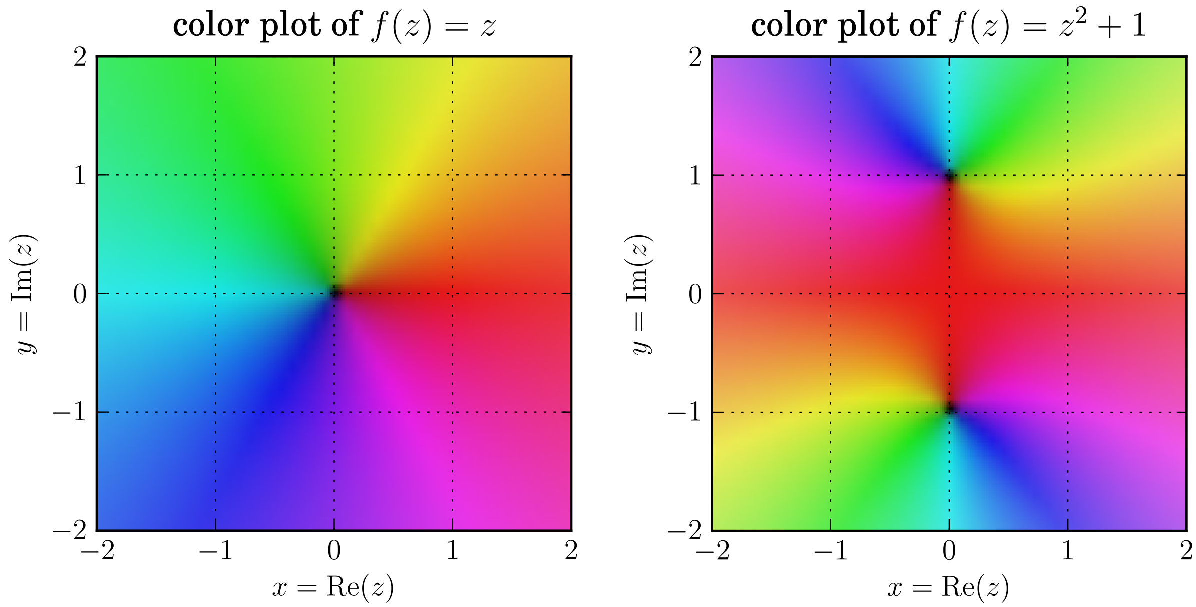
3.1.1 Visualizing Complex Functions

With a real function $f(x)$ we can just draw a two-dimensional graph with x on one axis and $f(x)$ on the other. With a complex function of a complex argument, this is not so easy, since we'd need to draw a surface in a four-dimensional space to show the u and v values corresponding to each x and y. Instead, we start with the complex plane, which is a two-dimensional space with coördinates x and y. Each point is a complex number $z = x+iy$, and we can then imagine the function f defining a complex number $f(z) = w = u(x, y) + iv(x, y)$ at each point. There are a few possibilities:

Use color: If we write the function in polar form

$$f(z) = \rho(x, y)e^{i\phi(x,y)} \tag{3.4}$$

then we can represent the value $f(z)$ by a color, where the modulus ρ determines how bright the color is and the argument ϕ determines the hue (red, green, cyan, yellow, etc., which is handy because hue is periodic around the color wheel just like the angular coördinate is periodic). So 0 would be black, real numbers would be either shades of red that got brighter and brighter as they got more positive or shades of cyan that got brighter and brighter as they got more negative, and in general complex numbers would get brighter and brighter as you went off away from zero in any direction. Here's an example of one such scheme:



There are lots of fine points, like whether it's really a good idea to use red for positive numbers $\mathrm{Arg}(z) = 0$, and the fact that we perceive some hues as being inherently brighter than others,[5] but you'll see these sorts of visualizations around the web.

Use vectors: Just as we can use the coördinates (x, y) to associate a number $z = x + iy$ with a point in the plane, we can think of u and v as the components of a vector in two dimensions, and if we recall that $u(x, y)$ and $v(x, y)$ gives us two numbers at each point, we can use the function $f(z)$ to construct a *vector field*, i.e., a prescription for defining a vector at each point. The most obvious choice is to use u as the x component and v as the real component, i.e., to define the vector field[6]

$$\vec{T}(x, y) = u(x, y)\,\hat{x} + v(x, y)\,\hat{y} \tag{3.5}$$

This is the vector field that Zill and Wright use throughout the book.

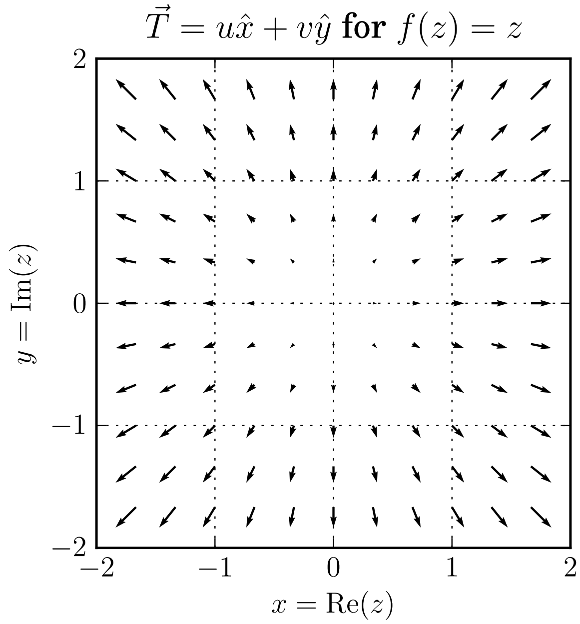
[5] Ask your friends majoring in Color Science!

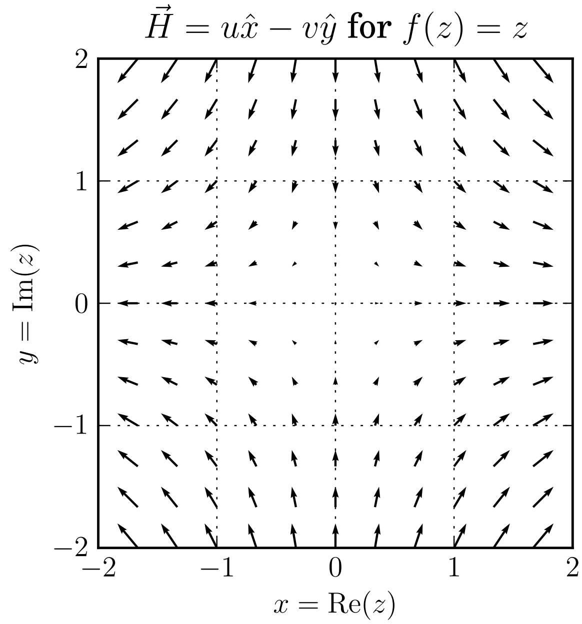
[6] You may have seen the unit vectors in the x and y directions written as $\hat{\imath}$ and $\hat{\jmath}$; Zill and Wright refer to them as **i** and **j**. To avoid confusion between the vector **i** and the complex number i, I'll call them $\hat{x}$ and $\hat{y}$.

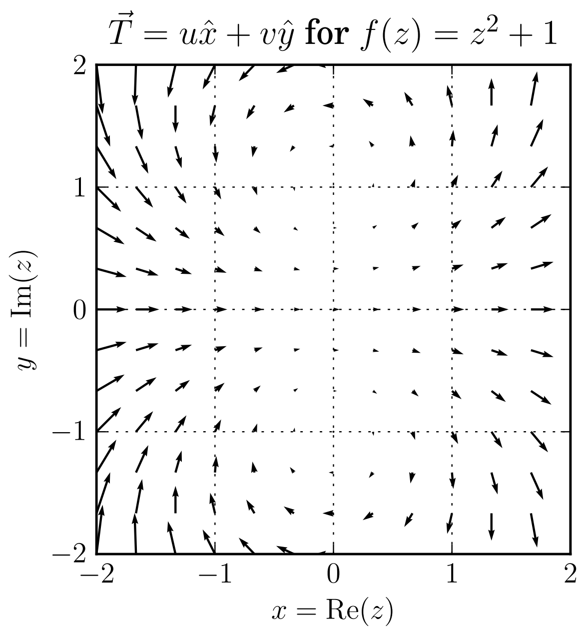
It turns out, though, that there's a different choice which makes some of the connections between complex analysis and vector calculus more straightforward. That is called the *Pólya vector field*

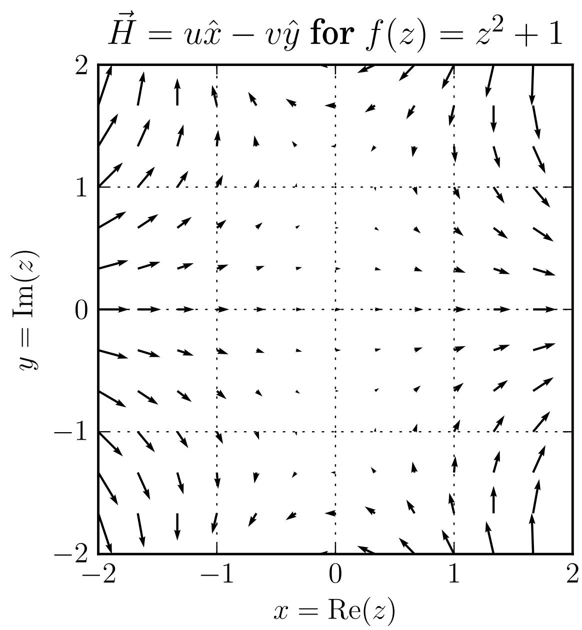
$$\vec{H}(x,y) = u(x,y)\,\hat{x} - v(x,y)\,\hat{y} \tag{3.6}$$

Here are some examples of both vector fields:









Map the z plane into the w plane: Finally, rather than trying to represent the complex number $w = f(z)$ at each point in the complex plane, we can imagine two planes, side by side, one representing z and the other representing w, and consider the function f as mapping the first plane onto the second plane. For instance, if $f(z) = z^2 + 1$, consider the line $x = 1$, which runs parallel to the imaginary axis. Each possible value of y corresponds to a point on this line. Using

$$u(x, y) = x^2 - y^2 + 1 \tag{3.3a}$$
$$v(x, y) = 2xy \tag{3.3b}$$

we note that the corresponding curve in the w plane is

$$u(1, y) = 2 - y^2 \tag{3.7a}$$
$$v(1, y) = 2y \tag{3.7b}$$

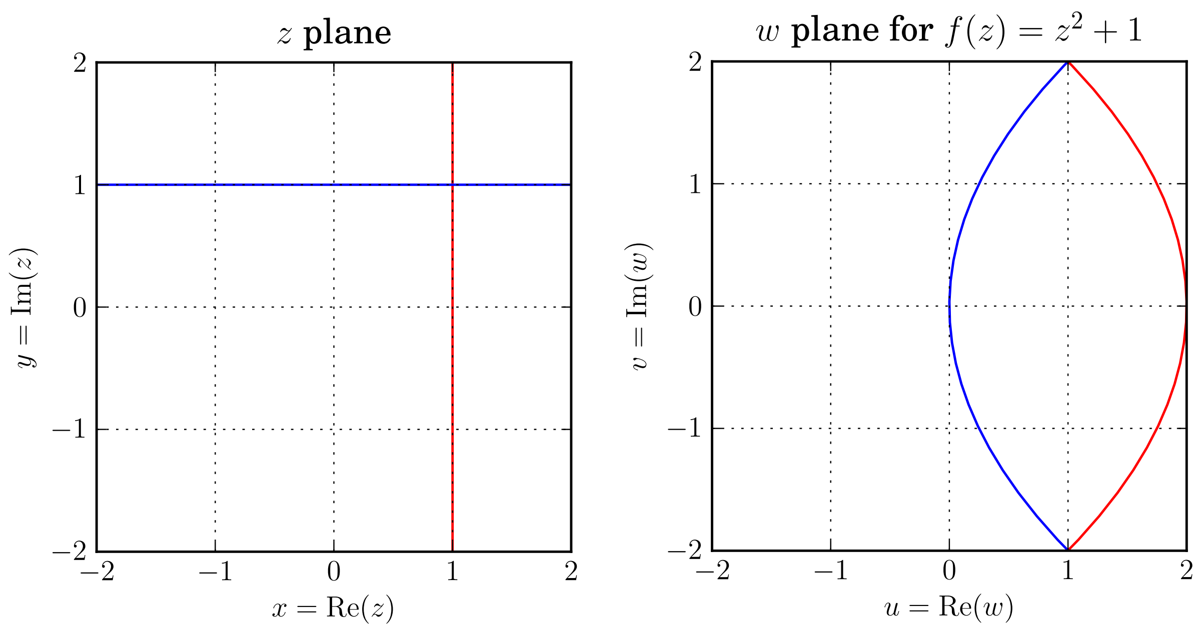
The curve, written in terms of u and v, is $u = 2 - v^2/4$, which describes a parabola.

On the other hand, the line $y = 1$ is parallel to the real axis, and its image is parameterized as

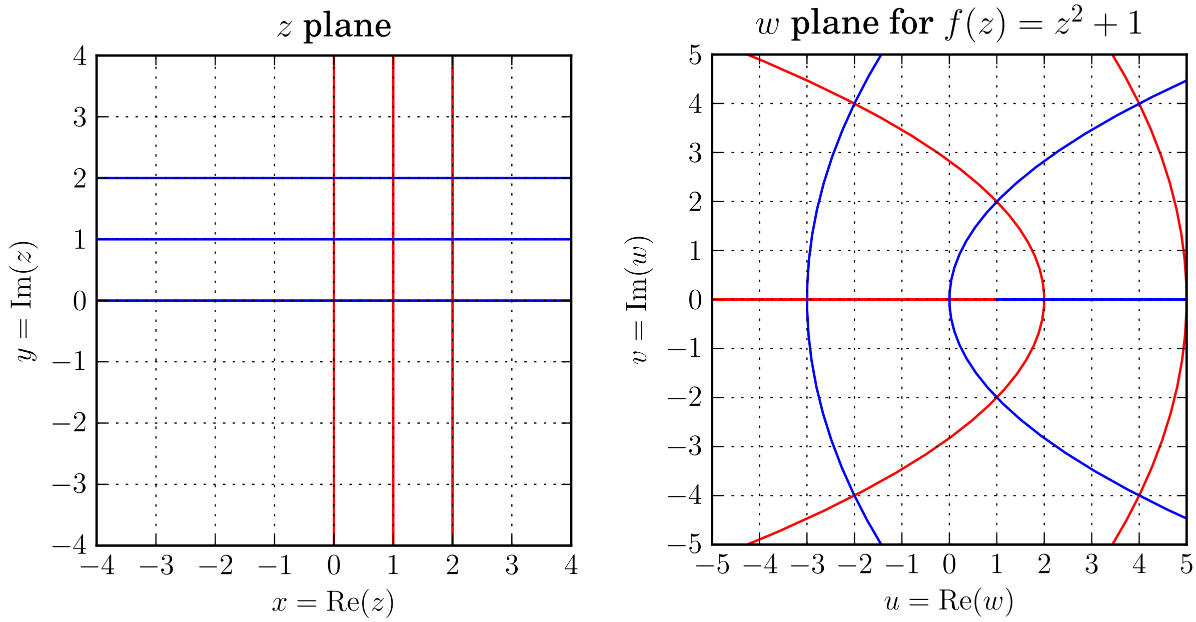
$$u(x, 1) = x^2 \tag{3.8a}$$
$$v(x, 1) = 2x \tag{3.8b}$$

which is the parabola $u = v^2/4$

We can plot these curves, the first in red and the second in blue:



We can continue this process; here I've added the curves $x = 0$, $x = 2$, $y = 0$ and $y = 1$ to the z and w planes:



3.2 Complex Differentiation

Recall that the derivative $f'(x) \equiv \frac{df}{dx}$ of a real function $f(x)$ is defined so that at a point $f(x_0)$ it has the value

$$f'(x_0) = \left.\frac{df}{dx}\right|_{x=x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \,. \tag{3.9}$$

We can make a similar definition for a complex function $f(z)$, that its definition

$$f'(z_0) = \left.\frac{df}{dz}\right|_{z=z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \,. \tag{3.10}$$

The tricky part is what it means to take the limit as $z - z_0$ goes to zero. Remember that, in real calculus, a limit statement like $\lim_{x \to x_0} g(x) = L$ means that you can always make $g(x)$ as close as you want to L by requiring that x be sufficiently close to x_0. This is written in so-called epsilon-and-delta notation as: "For each $\epsilon > 0$, there is a $\delta > 0$ such that $|g(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta$." Note that, in order for the limit to exist, we need to get the same L whether we approach x_0 from above ($x_0 < x < x_0 + \delta$) or below ($x_0 - \delta < x < x_0$).

In complex analysis, we make the same definition. The limit $\lim_{z \to z_0} g(z) = L$ (note L is now a complex number) means that you can always make $g(z)$ as close as you want to L by requiring that z be sufficiently close to z_0. Now "close" is defined using the modulus

$$|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2} \tag{3.11}$$

so $|z - z_0| < \delta$ is an open disk of radius δ. The formal definition of $\lim_{z \to z_0} g(z) = L$ is now "For each $\epsilon > 0$, there is a $\delta > 0$ such that $|g(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$." Note that z, z_0, L and $g(z)$ are all complex numbers, but ϵ and δ are real numbers. The tricky thing about this limit is that in order for it to exist, $g(z)$ needs to go to the same L no matter what direction z approaches z_0 from. This means that functions which have derivatives are more limited category than you would think.

An example:

$$\text{If } f(z) = z^2,\ f'(z_0) = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z + z_0)(z - z_0)}{z - z_0} = \lim_{z \to z_0} z + z_0 = 2z_0 \tag{3.12}$$

so $f'(z) = 2z$. If you go through the derivations of the various rules of differentiation, you'll find that they still hold for complex derivatives:

- Derivative of a complex constant c:
$$\frac{dc}{dz} = 0 \tag{3.13}$$
- Derivative of a power:
$$\frac{d}{dz} z^n = n z^{n-1} \tag{3.14}$$
- Multiplication by a complex constant c:
$$\frac{d}{dz}[c f(z)] = c f'(z) \tag{3.15}$$
- Sum rule:
$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z) \tag{3.16}$$
- Product rule:
$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z) \tag{3.17}$$
- Quotient rule:
$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2} \qquad \text{where } g(z) \neq 0 \tag{3.18}$$
- Chain rule:
$$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z) \tag{3.19}$$

  Note that the chain rule is a lot easier to remember if we let $w = g(z)$ and $\tau = f(w) = f(g(z))$. Then the chain rule is
$$\frac{d\tau}{dz} = \frac{d\tau}{dw}\frac{dw}{dz} \tag{3.20}$$

As an example, consider the function

$$f(z) = \frac{2z^2 + 1}{z + 1} \tag{3.21}$$

We can apply the rules of differentiation to say that, for $z \neq -1$,

$$\begin{aligned} f'(z) &= \frac{(z+1)\frac{d(2z^2+1)}{dz} - (2z^2+1)\frac{d(z+1)}{dz}}{4z^2} = \frac{(z+1)(4z) - (2z^2+1)}{(z+1)^2} \\ &= \frac{4z^2 + 4z - 2z^2 - 1}{(z+1)^2} = \frac{2z^2 + 4z - 1}{(z+1)^2} \end{aligned} \tag{3.22}$$

Finally, many simple-seeming functions do not actually have derivatives. Consider the complex conjugate $\bar{z} = x - iy$, treated as a function $f(z) = \bar{z}$. We want to write

$$f'(z_0) = \lim_{z_0 \to z} \frac{\bar{z} - \bar{z}_0}{z - z_0} \tag{3.23}$$

but we have to be able to have z approach z_0 from any direction. Let

$$z = z_0 + \rho e^{i\phi} \tag{3.24}$$

for some fixed ϕ and then take ρ to zero. Since

$$\bar{z} = \bar{z}_0 + \rho e^{-i\phi} \tag{3.25}$$

we have

$$\frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{\rho e^{i\phi}}{\rho e^{-i\phi}} = e^{i2\phi} \tag{3.26}$$

which will be different for different choices of ϕ. For instance, if we approach parallel to the real axis ($\phi = 0$ or π) we'll get 1, and if we approach parallel to the imaginary axis ($\phi = \pm\pi/2$) we'll get -1. So the function $f(z) = \bar{z}$ is not differentiable anywhere. This means that many functions that can be constructed using the complex conjugate, like $|z|$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, do not have well-defined derivatives either. Next time we'll see a straightforward way to check whether a function is differentiable.

Practice Problems

17.4.1, 17.4.5, 17.4.9, 17.4.13, 17.4.17, 17.4.21, 17.4.33, 17.4.37

Thursday 6 December 2012

3.3 The Cauchy-Riemann Equations

We saw last time that it only makes sense to talk about the derivative $f'(z)$ of a function

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) \tag{3.27}$$

at a point z_0 if the limit

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \tag{3.28}$$

exists, i.e., if you get the same answer when you let z approach z_0 from any direction. We'll now consider two different ways that z can approach z_0, and by requiring that the derivatives calculated in both ways match, obtain a requirement on $u(x,y)$ and $v(x,y)$ for $f'(z)$ to be defined.

First, consider approaching z_0 along a path parallel to the real axis, i.e., let $z = x + iy_0$ and consider taking $x \to x_0$ while the imaginary part remains y_0. Then the derivative is

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{x\to x_0} \frac{[u(x,y_0)+iv(x,y_0)]-[u(x_0,y_0)+iv(x_0,y_0)]}{(x+iy_0)-(x_0+iy_0)} \\ &= \lim_{x\to x_0} \frac{[u(x,y_0)-u(x_0,y_0)]+i[v(x,y_0)-v(x_0,y_0)]}{x-x_0} = u_{,x}(x_0,y_0) + iv_{,x}(x_0,y_0) \end{aligned} \tag{3.29}$$

where we have used the definition of the partial derivative, e.g.,

$$u_{,x}(x_0,y_0) = \left.\frac{\partial u}{\partial x}\right|_{(x,y)=(x_0,y_0)} = \lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} \tag{3.30}$$

So

$$\text{If } f(z) \text{ is differentiable, then } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{3.31}$$

On the other hand, we can approach z_0 from another direction, parallel to the imaginary axis, by letting $z = x_0 + iy$ and considering the limit as $y \to y_0$ while the real part remains x_0. Then the derivative is

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{y\to y_0} \frac{[u(x_0,y)+iv(x_0,y)]-[u(x_0,y_0)+iv(x_0,y_0)]}{(x_0+iy)-(x_0+iy_0)} \\ &= \lim_{y\to y_0} \frac{[u(x_0,y)-u(x_0,y_0)]+i[v(x_0,y)-v(x_0,y_0)]}{i(y-y_0)} = -iu_{,y}(x_0,y_0) + v_{,y}(x_0,y_0) \end{aligned} \tag{3.32}$$

which means that

$$\text{If } f(z) \text{ is differentiable, then } f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{3.33}$$

In order for the derivative to be well-defined, (3.31) and (3.33) have to agree, i.e.,

$$\text{If } f(z) \text{ is differentiable, then } \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{3.34}$$

Splitting this equation into its real and imaginary parts gives us the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{3.35a}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{3.35b}$$

These equations have to be satisfied at any point where $f(z)$ is differentiable. This means they are a *necessary* condition for differentiability. We won't show it right now, but it turns out that they are also *sufficient*, i.e., if the Cauchy-Riemann equations are satisfied, and all of the partial derivatives are continuous, the function is differentiable. (This is basically because if you know how to approach along the real direction and along the imaginary direction, you can describe the results of approaching along an arbitrary direction as a superposition of the two.)

3.3.1 The Fine Print: Differentiable vs Analytic

If the complex derivative $f'(z_0)$ exists, we say $f(z)$ is differentiable at z_0. A closely related term is *analytic*; we say that $f(z)$ is analytic at z_0 if it is differentiable in some open region ("domain") containing z_0. (That was the point of all of that neighborhood business in section 17.3.) The bottom line is that if the Cauchy-Riemann equations are satisfied in an open region, and all of the partial derivatives are continuous, the function is analytic in that region.

As an example of the difference between differentiable and analytic, consider the function

$$f(z) = [\mathrm{Re}(z)]^2 = x^2 \tag{3.36}$$

which has $u(x, y) = x^2$ and $v(x, y) = 0$. The partial derivatives are

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0 \tag{3.37a}$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = 0 \tag{3.37b}$$

And we see that the Cauchy-Riemann equations are satisfied if $x = 0$ but not otherwise. The line $x = 0$ is the imaginary axis, and it's just a line. So $f(z)$ is differentiable along that line, but since any neighborhood contains some points not on that line, it is not analytic anywhere.

3.3.2 Aside: Vector Calculus

If we write the Cauchy-Riemann equations (3.35) as

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{3.38a}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{3.38b}$$

they should look kind of reminiscent of derivatives from vector calculus.

In two dimensions there are two interesting derivatives you can take of a vector field

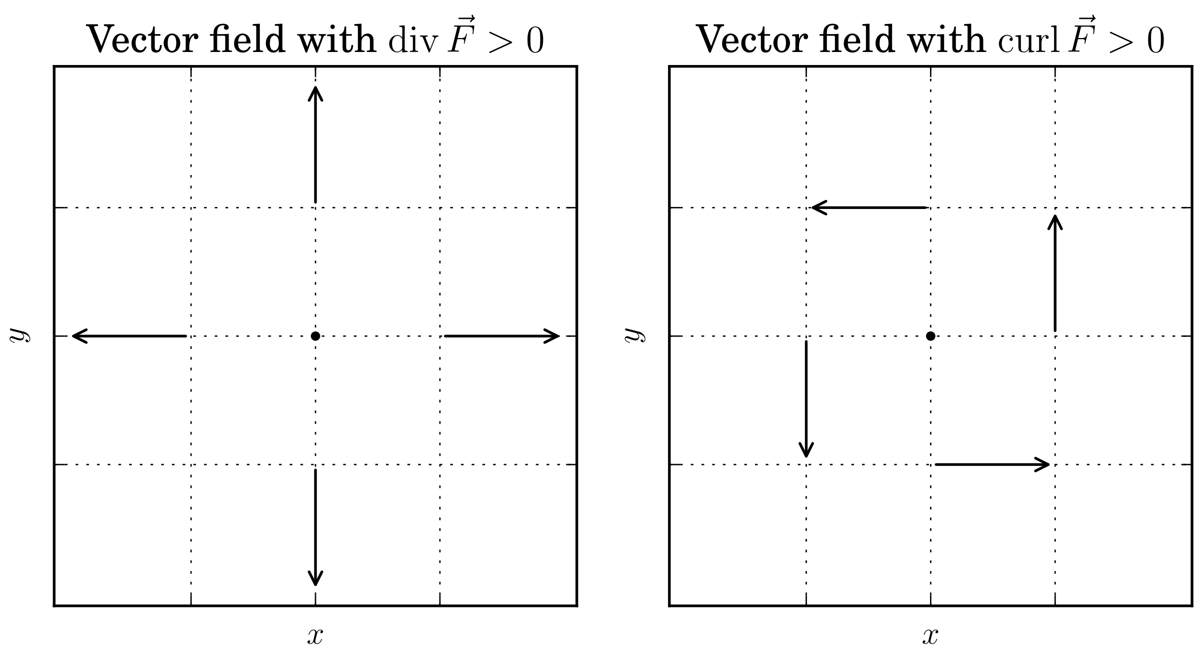
$$\vec{F}(x,y) = F_x(x,y)\,\hat{x} + F_y(x,y)\,\hat{y}\ . \tag{3.39}$$

(Note that F_x and F_y are components of $\vec{F}$, not partial derivatives.) The two derivatives are the divergence and the curl:

$$\operatorname{div}\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \tag{3.40a}$$

$$\operatorname{curl}\vec{F} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \tag{3.40b}$$

In three dimensions the divergence $\operatorname{div}\vec{F}$ is the scalar field $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$, and the curl $\operatorname{curl}\vec{F}$ is the vector field $\vec{\nabla}\times\vec{F}$, but in two dimensions, you can just think of each of them as producing a scalar field, i.e., a number for each point in the plane. We can see the geometrical meaning if we sketch vector fields with locally positive divergence or curl:



The vector field on the left, which has F_x increasing with x and F_y increasing with y, is "diverging" out from the point in the middle; if we calculated the outward flux of the vector field through a small bubble around that point, it would be positive.

The vector field on the right, which has F_y increasing with x and F_x decreasing with y, is "curling" counter-clockwise around the point in the middle; if we calculated the counter-clockwise circulation of the vector field around a small curve encircling that point, it would be positive.

Now consider the Pólya vector field

$$\vec{H} = u(x,y)\,\hat{x} - v(x,y)\hat{y} \tag{3.41}$$

which we defined last time. Its divergence and curl are

$$\operatorname{div} \vec{H} = \frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \tag{3.42a}$$

$$\operatorname{curl} \vec{H} = \frac{\partial u}{\partial y} - \frac{\partial(-v)}{\partial x} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.42b}$$

but these are exactly the expressions in (3.38), which vanish when the Cauchy-Riemann equations are satisfied. This means that **the Cauchy-Riemann equations for a function are equivalent to that function's Pólya vector field having zero divergence and zero curl.**[7]

3.3.3 Harmonic Functions

The Cauchy-Riemann equations are a pair of conditions on the two functions $u(x, y)$ and $v(x, y)$ together, but they also imply a separate differential equation for $u(x, y)$, and another one for $v(x, y)$. If we take the partial derivative of the first equation with respect to x and of the second equation with respect to y, we get

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \tag{3.43a}$$

$$\frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} \tag{3.43b}$$

Now, it is a property of partial derivatives that, if the functions involved have continuous partial derivatives, the partial derivatives commute. In this case,

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \tag{3.44}$$

This means that the two expressions in (3.43) are equal to each other, and

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \tag{3.45}$$

I.e., $u(x, y)$ solves *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3.46}$$

We say that $u(x, y)$ is a *harmonic function.* We can go through the same argument with the derivatives switched, and from

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial y} \tag{3.47a}$$

$$\frac{\partial}{\partial x}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} \tag{3.47b}$$

[7] This is one of the reasons the Pólya vector field $\vec{H} = u(x, y)\,\hat{x} - v(x, y)\hat{y}$ is more convenient than the field $\vec{T} = u(x, y)\,\hat{x} + v(x, y)\hat{y}$ which Zill and Wright use. The divergence and curl of $\vec{T}$ vanish if and only if the complex conjugate function $\bar{f}(z) = u(x, y) - iv(x, y)$ satisfies the Cauchy-Riemann equations.

deduce

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{3.48}$$

That means that **if** $f(z) = u(x, y) + iv(x, y)$ **is an analytic function, the functions** $u(x, y)$ **and** $v(x, y)$ **are both harmonic.** We refer to $v(x, y)$ as the conjugate harmonic function of $u(x, y)$ and vice versa.

If we start with a harmonic function $u(x, y)$, we can deduce its conjugate harmonic function $v(x, y)$, up to a constant, by imposing the Cauchy-Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{3.49a}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \tag{3.49b}$$

Example: Let $u(x, y) = x^2 - y^2 + 2x$. We can check that it is harmonic by calculating

$$\frac{\partial u}{\partial x} = 2x + 2 \qquad \frac{\partial^2 u}{\partial x^2} = 2 \tag{3.50a}$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial^2 u}{\partial y^2} = -2 \tag{3.50b}$$

$$\tag{3.50c}$$

so we do indeed have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. We can find y from the differential equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \tag{3.51a}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x + 2 \tag{3.51b}$$

Integrating (3.51a) with respect to x gives

$$v(x, y) = \int 2y\, dx = 2xy + h(y) \; ; \tag{3.52}$$

The integration "constant" is actually an arbitrary function of y alone, because such a function acts like a constant when we take the partial derivative with respect to x. Plugging (3.52) into (3.51b) gives us

$$\frac{\partial v}{\partial y} = 2x + h'(y) = 2x + 2 \tag{3.53}$$

which means that

$$h'(y) = 2 \tag{3.54}$$

this means

$$h(y) = 2y + C \tag{3.55}$$

where now C really is a constant, so the conjugate harmonic function of $u(x, y) = x^2 - y^2 + 2x$ is

$$v(x, y) = 2xy + 2y + C \tag{3.56}$$

The function is

$$f(z) = x^2 - y^2 + 2x + i(2xy + 2y + C) \tag{3.57}$$

The conjugate harmonic function is not just a handy way of taking one harmonic function and generating another; the functions $u(x, y)$ and $v(x, y)$ also have level surfaces ($u(x, y) = \text{const}$ and $v(x, y) = \text{const}$) which meet at right angles.

Practice Problems

17.5.1, 17.5.3, 17.5.9, 17.5.15, 17.5.17, 17.5.23, 17.5.25, 17.5.29, 17.5.32

Tuesday 11 December 2012

3.4 Some Specific Functions

The various rules of differentiation (sum, product, quotient, etc) mean that we can build up analytic functions from polynomials; for example, if n is a positive integer, z^n is analytic everywhere (with derivative nz^{n-1}) and z^{-n} is analytic (with derivative $-nz^{-(n+1)}$ everywhere except at $z = 0$. We now consider the behavior of some transcendental functions on the complex plane.

3.4.1 Exponentials, Logarithms and Powers

We already know how to take the exponential e^x of a real number x, and Euler's formula lets us take the exponential $e^{iy} = \cos y + i \sin y$ of an imaginary number iy. So for a general complex number $z = x + iy$, the exponential function is

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y \tag{3.58}$$

We can check that it's analytic by verifying that

$$u(x, y) = e^x \cos y \tag{3.59a}$$
$$v(x, y) = e^x \sin y \tag{3.59b}$$

satisfy the Cauchy-Riemann equations. The partial derivatives are

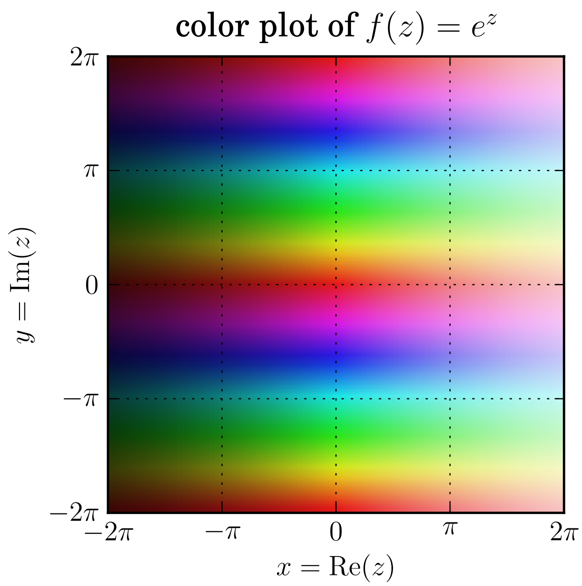
$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y \tag{3.60a}$$
$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y \tag{3.60b}$$

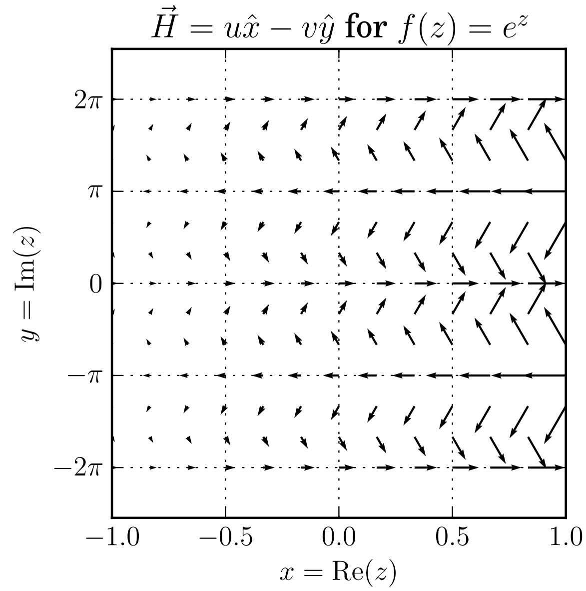
from which we can see that indeed $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Note that the exponential function is periodic in the complex direction:

$$e^{z+i2n\pi} = e^z \qquad n \in \mathbb{Z} \tag{3.61}$$

This can be seen in a color plot of the exponential function:





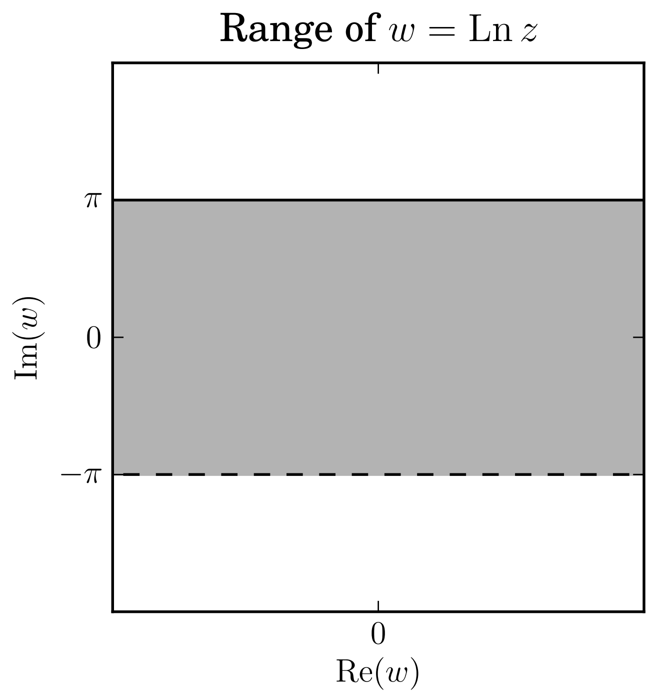
The Polya vector field for the exponential field is simple conceptually: the vectors have a length which grows with increasing x and points in a direction deterimined by y. It's actually hard to draw to scale, though, since the exponential e^x grows so fast. For example. if $e^0 = 1$ is a sensible length, $e^{\pi} \approx 23.14$ is inconveniently long and $e^{\pi} \approx 0.04321$ is too small to see easily. So in the plot above, the scales on the x and y axes are different, which makes it hard to get more than a qualitative idea. (Still, it's useful to stare at it and convince yourself that div $\vec{H}$ and curl $\vec{H}$ are both zero, as they must be for an analytic function.)

The natural logarithm is the inverse of the exponential function. This is complicated by the fact that the function e^z is not one-to-one, i.e., there are many values of w such that $z = e^w$. For example, $i = e^{i\pi/2} = e^{i3\pi/2} = e^{-i\pi/2}$ etc. This is the same issue we have with roots like $z^{1/5}$. To talk about the specific value of the logarithm, we need to spell out some notation; here are the names given by Zill and Wright for the different natural logarithms:

- $\log_e$ is the natural logarithm as a real number; it is a one-to-one and onto function which takes positive real numbers $(0, \infty)$ to real numbers $(-\infty, \infty)$. It is the inverse of the exponential, so $e^{\log_e(x)} = x$ and $\log_e(e^x) = x$.

- $\ln z$ is a multi-valued complex "function", defined so that $w = \ln z$ if w is any number such that $z = e^w$.

- $\text{Ln}\, z$ is the principal value of the natural logarithm, which is the inverse of the exponential, with $w = \text{Ln}\, z$ restricted to lie in the region $-\pi < \text{Ln}\, w < \pi$



To work out the possible values of the multi-valued natural logarithm $w = u+iv = \ln z = \ln(x+iy) = \ln(re^{i\theta})$ we require that $e^w = z$, i.e.,

$$re^{i\theta} = x + iy = z = e^w = e^{u+iv} = e^u e^{iv} = e^u \cos v + ie^u \sin v \tag{3.62}$$

For this to be satisfied, we need

$$x = e^u \cos v \tag{3.63a}$$

$$y = e^u \sin v \tag{3.63b}$$

or equivalently

$$r = e^u \tag{3.64a}$$

$$e^{i\theta} = e^{iv} \tag{3.64b}$$

The first equation relates two positive real numbers, so we can take the natural logarithm and find

$$u = \log_e r = \log_e \sqrt{x^2 + y^2}\ ; \tag{3.65}$$

The second equation says that v can be any number such that

$$\cos v = \cos\theta = \frac{x}{r} \tag{3.66a}$$

$$\sin v = \sin\theta = \frac{y}{r} \tag{3.66b}$$

which means v can be θ plus any integer multiple of 2π. But that's just the multi-valued argument of z

$$\arg z = \texttt{atan2}(y, x) + n2\pi \qquad n \in \mathbb{Z} \tag{3.67}$$

This means the multi-valued natural logarithm is

$$\ln z = \log_e |z| + i \arg z \tag{3.68}$$

The principal value just comes from taking the principal value of the argument:

$$\operatorname{Ln} z = \log_e |z| + i \operatorname{Arg} z \tag{3.69}$$

The usual nice features of the logarithm apply to the multi-valued complex logarithm, e.g.,

$$\ln(z_1 z_2) = \ln z_1 + \ln z_1 \tag{3.70a}$$

$$\ln \frac{z_1}{z_2} = \ln z_1 - \ln z_1 \tag{3.70b}$$

Again, this may not hold for the principal values, e.g., let $z_1 = -1 + i$ and $z_2 = i$. Then $|z_1| = \sqrt{2}$, $\operatorname{Arg}(z_1) = \texttt{atan2}(1, -1) = 3\pi/4$, $|z_2| = 1$, $\operatorname{Arg}(z_2) = \texttt{atan2}(1, 0) = \pi/2$. So

$$\operatorname{Ln} z_1 = \operatorname{Ln}(1 + i) = \log_e(\sqrt{2}) + i\frac{3\pi}{4} = \frac{1}{2}\log_e 2 + i\frac{3\pi}{4} \tag{3.71a}$$

$$\operatorname{Ln} z_2 = \operatorname{Ln}(i) = \log_e(1) + i\frac{\pi}{2} = i\frac{\pi}{2} \tag{3.71b}$$

On the other hand, $z_1 z_2 = (-1 + i)(i) = -1 - i$, so $|z_1 z_2| = \sqrt{2}$, and $\operatorname{Arg}(z_1 z_2)\,\texttt{atan2}(-1, -1) = -3\pi/4$. This means that

$$\operatorname{Ln}(z_1 z_2) = \operatorname{Ln}(-1 - i) = \log_e(\sqrt{2}) - i\frac{3\pi}{4} = \frac{1}{2}\log_e 2 - i\frac{3\pi}{4} \tag{3.72}$$

On the other hand,

$$\operatorname{Ln} z_1 + \operatorname{Ln} z_2 = \log_e(\sqrt{2}) + i\frac{3\pi}{4} + i\pi = \frac{1}{2}\log_e 2 + i\frac{5\pi}{4} \tag{3.73}$$

so we see they differ by a multiple of 2π.

If consider $\operatorname{Ln} z$ as a complex function

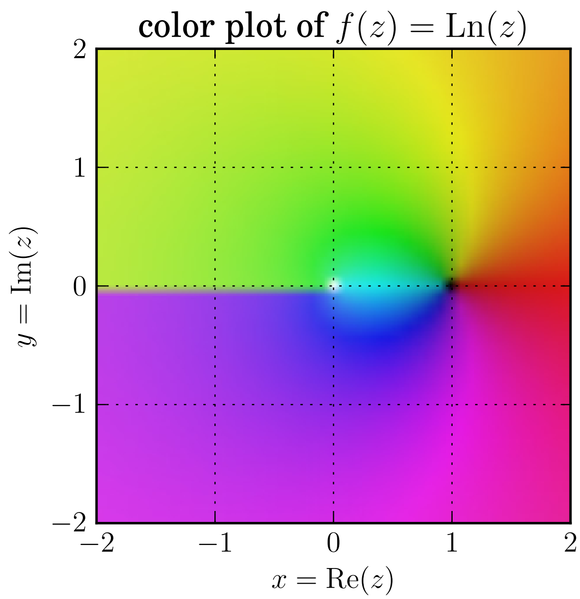
$$\operatorname{Ln} z = \operatorname{Ln}(x + iy) = \log_e |z| + i \operatorname{Arg} z = \frac{1}{2}\log_e(x^2 + y^2) + i\,\texttt{atan2}(y, x) \tag{3.74}$$

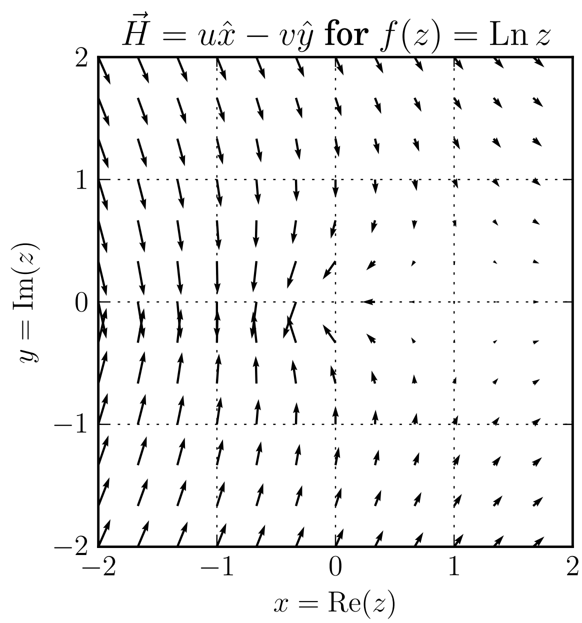
then $f(z) = \operatorname{Ln} z = u(x, y) + iv(x, y)$ has real and imaginary parts

$$u(x, y) = \frac{1}{2}\log_e(x^2 + y^2) \tag{3.75a}$$

$$v(x, y) = \texttt{atan2}(y, x) = \operatorname{Arg}(x + iy) \tag{3.75b}$$

We see that both $u(x, y)$ and $v(x, y)$ are undefined at the origin $(x, y) = (0, 0)$. They're defined everywhere else, but $v(x, y)$ is discontinuous on the negative x axis, since $\texttt{atan2}(y, x)$ is defined to be π if $x < 0$ and $y = 0$, but it approaches $-\pi$ as we approach the negative x axis from the fourth quadrant. So $\operatorname{Ln} z$ is undefined at $z = 0$ and it's discontinuous on the negative real axis, but it's continuous everywhere else:





You can also show that it's analytic on that region[8], but we'll just show what the derivative must be, by implicit differentiation: If $w = \mathrm{Ln}\, z$, so that $z = e^w$, then, by the chain rule,

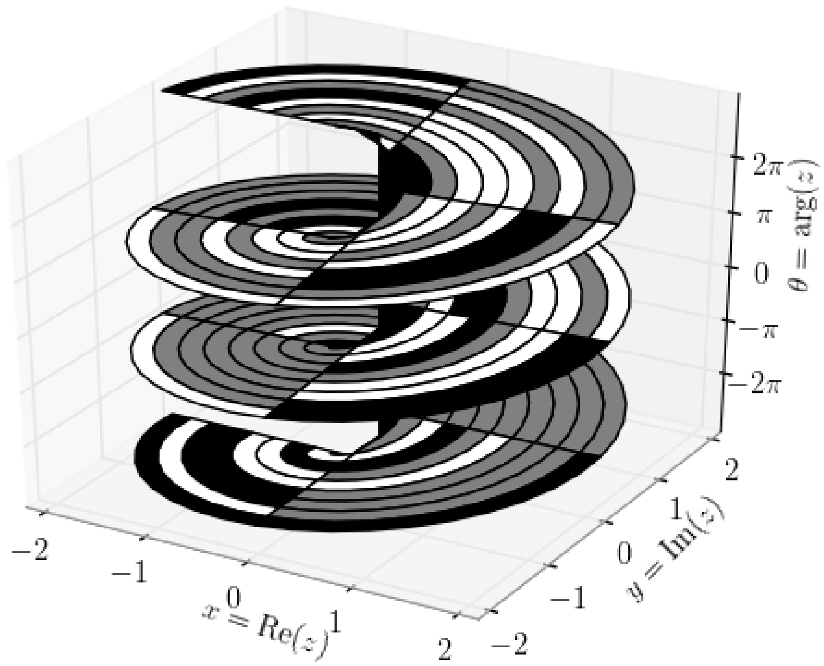
$$1 = e^w \frac{dw}{dz} = z \frac{dw}{dz} \tag{3.76}$$

but dividing by z gives

$$\frac{dw}{dz} = \frac{d}{dz} \mathrm{Ln}\, z = \frac{1}{z} \qquad \text{unless } z \text{ is non-positive real number} \tag{3.77}$$

Note that as we move around the complex plane, the multi-valued $\ln z$, like the multi-valued $\arg(z)$, is continuous everywhere except for the origin $z = 0$, but if we go once counter-clockwise around the origin, $\arg z$ has increased by 2π, and $\ln z$ has increased by $i2\pi$, from where we started. We can visualize a "parking ramp" where going around the origin moves us smoothly up to another level:

[8] You have to consider the derivatives of `atan2`(y, x), which are basically the derivatives of the polar coördinate θ with respect to x and y.



Choosing a single principal value for $\operatorname{Ln} z$ or $\operatorname{Arg} z$ means picking one "branch" of this surface, and imposing a discontinuity called a "branch cut". Our choice is to put the branch cut along the negative real axis, but we could have made a different choice, like requiring $0 \leq \theta < 2\pi$.

Complex powers: Finally, note that the complex logarithm lets us talk about raising a complex number to an arbitrary complex power. Recall that if x is a positive real number and a is any real number, taking the exponential of both sides of

$$\log_e(x^a) = a \log_e x \tag{3.78}$$

allows us to write

$$x^a = e^{a \log_e x} \; ; \tag{3.79}$$

for complex numbers z and α, we define

$$z^\alpha = e^{\alpha \ln z} \tag{3.80}$$

In general, since

$$\ln z = \log_e |z| + i \arg z = \log_e |z| + i \operatorname{Arg} z + i\, k2\pi \qquad k \in \mathbb{Z} \tag{3.81}$$

there are an infinite number of values for z^α:

$$z^\alpha = \exp\left[\alpha \log_e |z| + i\, \alpha(\operatorname{Arg} z + k2\pi)\right] \tag{3.82}$$

Note that in the special case where $\alpha = n$, a real integer, we get

$$z^n = \exp\left[n \log_e |z| + i\, n(\operatorname{Arg} z + k2\pi)\right] = |z|^n \; e^{-i\,n \operatorname{Arg} z} e^{i\,nk2\pi} = |z|^n \; e^{-i\,n \operatorname{Arg} z} \tag{3.83}$$

which is a single value, because $e^{i\,nk2\pi} = 1$. When $\alpha = m/n$, a rational number with m and n being integers with no non-trivial common factor, we get a similar case to what we had with roots:

$$z^{m/n} = \exp\left[\frac{m}{n} \log_e |z| + i\,\frac{m}{n}(\operatorname{Arg} z + k2\pi)\right] = \sqrt{n}\,|z|^m \, e^{i\,m \operatorname{Arg}(z)/n} e^{i\,2\pi mk/n} \tag{3.84}$$

the last factor, $e^{i\,2\pi mk/n}$ takes on different values for $k = 0, 1, \ldots (n-1)$, but it starts repeating itself for $k \geq n$ or $k < 0$, which means there are n unique values for $z^{m/n}$. This is also true for real numbers.

Example: what are the different values of $64^{1/6}$? Well, since $64 = 2^6$, and $\operatorname{Arg}(64) = 0$, we get

$$1^{1/6} = 2e^{i\,2\pi k/6} = 2e^{i\,k\pi/3} \qquad k = 0, 1, 2, 3, 4, 5 \tag{3.85}$$

working out the trig functions gives the six roots

$$2e^{i0} = 2 \tag{3.86a}$$

$$2e^{i\pi/3} = 2\cos\frac{\pi}{3} + i2\sin\frac{\pi}{3} = 1 + i\sqrt{3} \tag{3.86b}$$

$$2e^{i2\pi/3} = 2\cos\frac{2\pi}{3} + i2\sin\frac{2\pi}{3} = -1 + i\sqrt{3} \tag{3.86c}$$

$$2e^{i\pi} = -2 \tag{3.86d}$$

$$2e^{i4\pi/3} = 2\cos\frac{4\pi}{3} + i2\sin\frac{4\pi}{3} = -1 - i\sqrt{3} \tag{3.86e}$$

$$2e^{i5\pi/3} = 2\cos\frac{5\pi}{3} + i2\sin\frac{5\pi}{3} = 1 - i\sqrt{3} \tag{3.86f}$$

Practice Problems

17.6.3, 17.6.7, 17.6.15, 17.6.23, 17.6.25, 17.6.33, 17.6.35, 17.6.41

Thursday 13 December 2012

3.4.2 Trigonometric and Hyperbolic Functions

If we recall the Euler relation

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3.87}$$

and also

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \tag{3.88}$$

we see that the sine and cosine of a real number θ can be written

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{3.89a}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{3.89b}$$

From which we have

$$\tan\theta = \frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}} \tag{3.90}$$

and similar expressions for $\sec\theta$, $\csc\theta$, and $\cot\theta$. These relations are also the inspiration for the definition of the hyperbolic trig functions

$$\cosh\eta = \frac{e^{\eta} + e^{-\eta}}{2} \tag{3.91a}$$

$$\sinh\eta = \frac{e^{\eta} - e^{-\eta}}{2} \tag{3.91b}$$

with analogous definitions for $\tanh\eta$, $\operatorname{sech}\eta$, etc.

These definitions can be extended to complex arguments, i.e.,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{3.92a}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{3.92b}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \tag{3.92c}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \tag{3.92d}$$

Since $e^{\alpha z}$ is analytic over the whole complex plane (we say $e^{\alpha z}$ is entire), the same is true for $\cos z$, $\sin z$, $\cosh z$ and $\cosh z$. When we define other trigonometric and hyperbolic functions like $\sec z = \frac{1}{\cos z}$ and $\tanh z = \frac{\sinh z}{\cosh z}$, they'll be analytic except where their denominators go to zero. So it's useful to sort out the zeros of $\sin z$ and $\cos z$.

We know that the real values at which $\sin z = 0$ are integer multiples of π, and the real values at which $\cos z = 0$ are odd half-integer multiples of π, but are there any that are not on the real axis? If we write $\sin z$ and $\cos z$ in the form $f(z) = u(x, y) + iv(x, y)$ we get

$$\begin{aligned}\cos z = \cos(x + iy) &= \frac{1}{2}\left(e^{ix}e^{-y} + e^{-ix}e^{y}\right) = \frac{1}{2}\left(\cos x\, e^{-y} + i\sin x\, e^{-y} + \cos x\, e^{y} - i\sin x\, e^{y}\right) \\ &= \cos x\cosh y - i\sin x\sinh y\end{aligned} \tag{3.93}$$

A similar calculation shows

$$\sin z = \sin x\cosh y + i\cos x\sinh y \tag{3.94}$$

Now, to get $\sin z = 0$, we have to have $\sin x\cosh y = 0$ and $\cos x\sinh y = 0$. Since $\cosh y \geq 1$ for any real y, we need to have $\sin x = 0$, which means $x = n\pi$. In that case, $\cos x = (-1)^n \neq 0$, which means we can only get $\cos x \sinh y = 0$ if $\sinh y = 0$, i.e., $y = 0$. That means the only zeros of $\sin z$ are the real ones:

$$\sin z = 0 \qquad \text{iff } z \in \{n\pi | n \in \mathbb{Z}\} \tag{3.95}$$

a similar argument shows that

$$\cos z = 0 \qquad \text{iff } z \in \left\{ \left(n + \frac{1}{2} \right) \pi \middle| n \in \mathbb{Z} \right\} \tag{3.96}$$

To get the zeros of the hyperbolic sine and cosine, it helps to note that

$$\cos(iz) = \frac{e^{-z} + e^{z}}{2} = \cosh z \tag{3.97}$$

and

$$\sin(iz) = \frac{e^{-z} - e^{z}}{2i} = i \sinh z \tag{3.98}$$

This means that

$$\cosh z = \cos(-y + ix) = \cosh x \cos y + i \sinh x \sin y \tag{3.99a}$$

$$\sinh z = -i \sin(-y + ix) = \sinh x \cos y + i \cosh x \sin y \tag{3.99b}$$

so

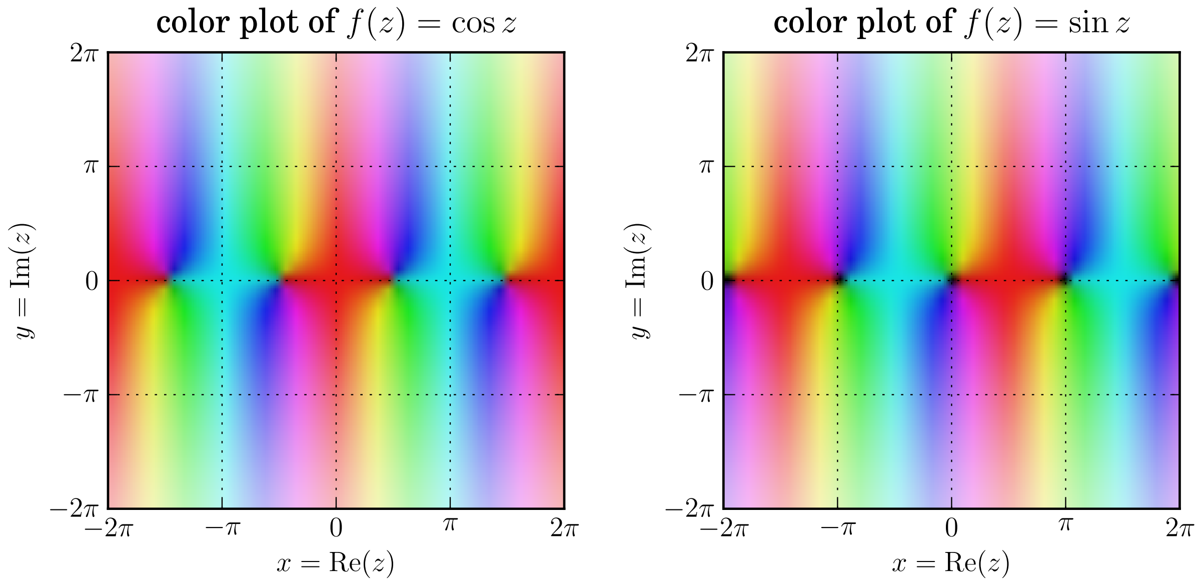
$$\sinh z = 0 \qquad \text{iff } z \in \{i\, n\pi | n \in \mathbb{Z}\} \tag{3.100}$$

a similar argument shows that

$$\cosh z = 0 \qquad \text{iff } z \in \left\{ i \left(n + \frac{1}{2} \right) \pi \middle| n \in \mathbb{Z} \right\} \tag{3.101}$$

so the zeros of $\cosh z$ and $\sinh z$ are all along the imaginary axis.

The trigonometric functions are periodic in the real direction with period 2π, e.g., $\sin(z+2\pi) = \sin(z)$:



Note that

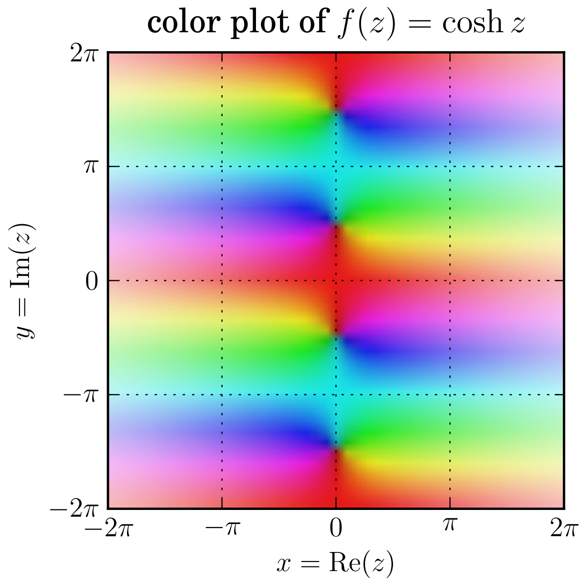
$$\sin\left(z + \frac{\pi}{2}\right) = \frac{e^{iz}e^{i\pi/2} - e^{-iz}e^{-i\pi/2}}{2i} = \frac{ie^{iz} - (-i)e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z \tag{3.102}$$

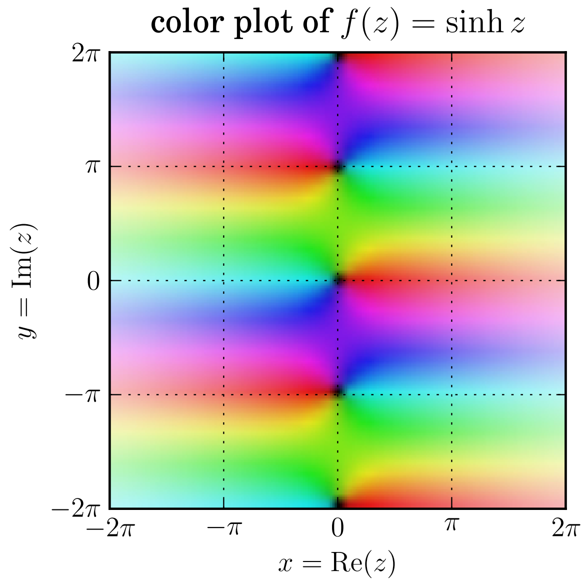
just as for real arguments. In fact the angle sum formulas still hold:

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \tag{3.103a}$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \tag{3.103b}$$

The hyperbolic functions are periodic in the imaginary direction, also with period 2π, e.g., $\cosh(z + 2\pi) = \cosh(z)$:





Notice that since

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 \tag{3.104a}$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \tag{3.104b}$$

we have

$$\sinh\left(z + i\frac{\pi}{2}\right) = \sinh z \cosh\left(i\frac{\pi}{2}\right) + \cosh z \sinh\left(i\frac{\pi}{2}\right) = \sinh z \cos\frac{\pi}{2} + i \cosh z \sin\frac{\pi}{2} = i \cosh z \tag{3.105}$$

Practice Problems

17.7.5, 17.7.7, 17.7.11, 17.7.15, 17.7.19, 17.7.21, 17.7.29, 17.7.30