Integration in the Complex Plane
(Zill & Wright Chapter 18)

1016-420-02: Complex Variables*

Winter 2012-2013

Contents

Tuesday 18 December 2012

1 Contour Integrals

1.1 Definition and Properties

Recall the definition of the definite integral

$$\int_{x_I}^{x_F} f(x)\,dx = \lim_{\Delta x_k \to 0} \sum_k f(x_k)\,\Delta x_k \tag{1.1}$$

We'd like to define a similar concept, integrating a function $f(z)$ from some point z_I to another point z_F. The problem is that, since z_I and z_F are points in the complex plane, there are different ways to get between them, and adding up the value of the function along one path will not give the same result as doing it along another path, even if they have the same endpoints. So instead we need to define a *contour integral*

$$\int_{\mathcal{C}} f(z)\,dz = \lim_{|\Delta z_k| \to 0} \sum_k f(z_k)\,\Delta z_k \tag{1.2}$$

which in general depends on the path $\mathcal{C}$ by which we choose to go from the initial point z_I to the final point z_F. To define such an integral, we *parametrize* the curve, i.e., write it as $z(t)$ where $z(t_I) = z_I$ and $z(t_F) = t_F$. (For example, if $\mathcal{C}$ is the semi-circle in the upper half-plane, from $z_I = 1$ to $z_F = -1$, we can use the angular polar coördinate as the parameter, and then $z(t) = e^{it}$ where $t_I = 0$ and $t_F = \pi$.) Given a parametrization of the curve, we define the contour integral as

$$\int_{\mathcal{C}} f(z)\,dz = \int_{t_I}^{t_F} f\bigl(z(t)\bigr)\,\frac{dz}{dt}\,dt = \int_{t_I}^{t_F} f\bigl(z(t)\bigr)\,z'(t)\,dt \tag{1.3}$$

You might worry that we could get a different result by choosing a different parametrization of the curve (e.g., suppose we'd chosen $z(t) = e^{it^2/\pi}$ in the example above), but you can show using the chain rule that the value of the integral doesn't change.

To specify more concretely the value of the integral (1.3), write $z(t) = x(t) + iy(t)$ and $f(x+iy) = u(x,y) + iv(x,y)$, where $x(t)$, $y(t)$, $u(x,y)$ and $v(x,y)$ are all real functions. Then

$$\begin{aligned}
\int_{\mathcal{C}} f(z)\,dz &= \int_{\mathcal{C}} \bigl[u(x,y) + iv(x,y)\bigr]\bigl[dx + idy\bigr] \\
&= \int_{t_I}^{t_F} \Bigl[u\bigl(x(t),y(t)\bigr) + iv\bigl(x(t),y(t)\bigr)\Bigr]\bigl[x'(t) + iy'(t)\bigr]\,dt \\
&= \int_{t_I}^{t_F} \Bigl[u\bigl(x(t),y(t)\bigr)x'(t) - v\bigl(x(t),y(t)\bigr)y'(t)\Bigr]\,dt \\
&\quad + i\int_{t_I}^{t_F} \Bigl[u\bigl(x(t),y(t)\bigr)y'(t) + v\bigl(x(t),y(t)\bigr)x'(t)\Bigr]\,dt
\end{aligned} \tag{1.4}$$

Note that Zill and Wright's equation (2), on page 797, which says sort of the same thing, really should be written with some parentheses, to indicate that e.g., $u\,dx$ and $v\,dy$ are both part of the first integral. I.e., it should read

$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} \bigl(u\,dx - v\,dy\bigr) + i \int_{\mathcal{C}} \bigl(v\,dx + u\,dy\bigr) \tag{1.5}$$

I would probably take points off on a test if it were written without the parentheses!

Because the contour integral is defined as the limit of a sum, it has the usual linearity properties of an integral, i.e., if α and β are complex constants,

$$\int_{\mathcal{C}} \bigl[\alpha\, f(z) + \beta\, g(z)\bigr]\,dz = \alpha \int_{\mathcal{C}} f(z)\,dz + \beta \int_{\mathcal{C}} g(z)\,dz \tag{1.6}$$

There are also contour equivalents of the properties of concatenation and reversal; just as

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \tag{1.7a}$$

and

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx\ , \tag{1.7b}$$

if $\mathcal{C}_1 + \mathcal{C}_2$ is the curve formed by chaining together $\mathcal{C}_1$ and $\mathcal{C}_2$ (which we can do if the final point of $\mathcal{C}_1$ is the initial point of $\mathcal{C}_2$), then

$$\int_{\mathcal{C}_1+\mathcal{C}_2} f(z)\,dz = \int_{\mathcal{C}_1} f(z)\,dz + \int_{\mathcal{C}_2} f(z)\,dz \tag{1.8}$$

and if $-\mathcal{C}$ is the curve we get by following $\mathcal{C}$ backwards (so the initial point of $-\mathcal{C}$ is the final point of $\mathcal{C}$ and vice versa), then

$$\int_{-\mathcal{C}} f(z)\,dz = -\int_{\mathcal{C}} f(z)\,dz \tag{1.9}$$

1.2 Evaluation

1.2.1 Example: $\int_{\mathcal{C}_1} \bar{z}\,dz$

Consider the example of $\int_{\mathcal{C}_1} \bar{z} dz$, where the function is $f(z) = \bar{z} = x - iy$, and the contour $\mathcal{C}_1$ is given by $x(t) = t$, $y(t) = t^2$, where $t_I = 0$ and $t_F = 1$. Then

$$dz = dx + idy = [x'(t) + iy'(t)]dt = (1 + i\,2t)dt \tag{1.10}$$

and

$$f(z(t)) = x(t) - iy(t) = t - i\,t^2 \tag{1.11}$$

so

$$\int_{\mathcal{C}_1} \bar{z}\, dz = \int_{t_I}^{t_F} f(z(t))z'(t)\, dt = \int_0^1 (t - i\, t^2)(1 + i\, 2t)\, dt = \int_0^1 (t + 2t^3)\, dt + i \int_0^1 (-t^2 + 2t^2)\, dt$$
$$= \left[\frac{t^2}{2} + \frac{t^4}{2}\right|_0^1 + i\, \frac{t^3}{3}\Big|_0^1 = \frac{1}{2} + \frac{1}{2} + i\, \frac{1}{3} = 1 + \frac{1}{3}\, i \tag{1.12}$$

1.2.2 Example: $\int_{\mathcal{C}_2} \bar{z}\, dz$

Now let's integrate the same integrand ($f(z) = \bar{z} = x - iy$) but along the curve $\mathcal{C}_2$ given by $x(t) = t$, $y(t) = t$, where $t_I = 0$ and $t_F = 1$. Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ both have the same endpoints $z_I = x(t_I) + i\, y(t_I) = 0$ and $z_F = x(t_F) + i\, y(t_F) = 1 + i$, but the paths are different. Now

$$dz = dx + idy = [x'(t) + iy'(t)]dt = (1 + i)dt \tag{1.13}$$

and

$$f(z(t)) = x(t) - iy(t) = t - i\, t \tag{1.14}$$

so

$$\int_{\mathcal{C}_1} \bar{z}\, dz = \int_{t_I}^{t_F} f(z(t))z'(t)\, dt = \int_0^1 (t - i\, t)(1 + i)\, dt = (1 - i)(1 + i) \int_0^1 t\, dt = 2 \int_0^1 t\, dt = t^2\Big|_0^1 = 1 \tag{1.15}$$

so $\int_{\mathcal{C}_1} \bar{z}\, dz \neq \int_{\mathcal{C}_2} \bar{z}\, dz$

1.2.3 Example: $\int_{\mathcal{C}} z^2\, dz$

As a more complicated contour, consider the integral of $f(z) = z^2$ along $\mathcal{C}$, which is a semicircle of radius 2, from $z_I = 2$ to $z_F = -2$, counter-clockwise in the upper half plane.

Now we have to start by making a parametrization of the curve. Different choices are possible, but a convenient one is

$$z = 2\, e^{it} \qquad \text{with} \qquad t_I = 0 \quad \text{and} \quad t_F = \pi \tag{1.16}$$

Rather than splitting things up into $u(x, y)$, $v(x, y)$, etc, it's more straightforward to work with $z(t)$ and write

$$dz = z'(t)\, dt = 2i\, e^{it}\, dt \tag{1.17}$$

and

$$f(z(t)) = (2e^{it})^2 = 4e^{i2t} \tag{1.18}$$

so

$$\int_{\mathcal{C}} \bar{z}\, dz = \int_{t_I}^{t_F} f(z(t))z'(t)\, dt = \int_0^\pi 4e^{i2t} 2i\, e^{it}\, dt = \int_0^\pi 8i\, e^{i3t}\, dt = \frac{8}{3}\, e^{i3t}\Big|_0^\pi = \frac{8}{3}\left(e^{i3\pi} - e^0\right)$$
$$= \frac{8}{3}(-1 - 1) = -\frac{16}{3} \tag{1.19}$$

1.3 The ML Limit

It is occasionally useful to place an upper limit on a contour integral, rather than evaluating it. A limit that can be placed on any contour integral of a continuous function along a smooth curve is the so-called ML limit:

$$\left| \int_{\mathcal{C}} f(z)\, dz \right| \leq ML \tag{1.20}$$

where M is the maximum value of $|f(z)|$:

$$|f(z)| \leq M \qquad \text{for all } z \text{ on } \mathcal{C} \tag{1.21}$$

and L is the length of $\mathcal{C}$.

1.4 Circulation and Flux

The contour integral also appears in vector calculus, where can evaluate the integral of a scalar field $\Psi(\vec{r})$ along some path $\vec{r}(t)$ as

$$\int_{\mathcal{C}} \Psi(\vec{r})\, ds = \int_{t_I}^{t_F} \Psi\left(\vec{r}(t)\right) \left| \frac{d\vec{r}}{dt} \right| dt \ , \tag{1.22}$$

where

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}} \ . \tag{1.23}$$

In two dimensions,

$$ds = \sqrt{(dx)^2 + (dy)^2} \tag{1.24}$$

i.e.,

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \tag{1.25}$$

Given a vector field

$$\vec{F}(\vec{r}) = \vec{F}(x,y) = F_x(x,y)\, \hat{x} + F_y(x,y)\, \hat{y} \ , \tag{1.26}$$

and a smooth closed curve $\mathcal{C}$, we're often interested in two contour integrals, known as the flux of $\vec{F}$ through $\mathcal{C}$ and the circulation of $\vec{F}$ around $\mathcal{C}$. At each point on the closed curve $\mathcal{C}$ we define a unit vector $\hat{t}$ pointing along the curve in the counter-clockwise direction and another unit vector $\hat{n}$ pointing perpendicular to the curve in the outward direction. Then the integrals of interest are

$$\text{Circulation of } \vec{F} \text{ around } \mathcal{C} = \oint_{\mathcal{C}} \vec{F} \cdot \hat{t}\, ds \tag{1.27a}$$

$$\text{Flux of } \vec{F} \text{ through } \mathcal{C} = \oint_{\mathcal{C}} \vec{F} \cdot \hat{n}\, ds \tag{1.27b}$$

where the arrow on the integral sign emphasizes that $\mathcal{C}$ is a closed curve traversed counter-clockwise. A little bit of geometry shows

$$\hat{t}\,ds = \hat{x}\,dx + \hat{y}\,dy \tag{1.28a}$$
$$\hat{n}\,ds = \hat{x}\,dy - \hat{y}\,dx \tag{1.28b}$$

so

$$\text{Circulation} = \oint_{\mathcal{C}} \vec{F}\cdot\hat{t}\,ds = \oint_{\mathcal{C}} [F_x\,dx + F_y\,dy] \tag{1.29a}$$
$$\text{Flux} = \oint_{\mathcal{C}} \vec{F}\cdot\hat{n}\,ds = \oint_{\mathcal{C}} [F_x\,dy - F_y\,dx] \tag{1.29b}$$

Now, consider our friend the Pólya vector field associated with a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, defined

$$\vec{H} = u(x,y)\,\hat{x} - v(x,y)\hat{y} \tag{1.30}$$

with the minus sign on the y component. Its circulation around and flux through some closed curve $\mathcal{C}$ are

$$\text{Circulation of } \vec{H} \text{ around } \mathcal{C} = \oint_{\mathcal{C}} [H_x\,dx + H_y\,dy] = \oint_{\mathcal{C}} [u\,dx - v\,dy] \tag{1.31a}$$
$$\text{Flux of } \vec{H} \text{ through } \mathcal{C} = \oint_{\mathcal{C}} [H_x\,dy - H_y\,dx] = \oint_{\mathcal{C}} [u\,dy + v\,dx] \tag{1.31b}$$

But if we compare this to (1.4) or (1.5), we see that these are just the real and imaginary parts of the contour integral $\oint_{\mathcal{C}} f(z)\,dz$! I.e.,

$$\text{Circulation of } \vec{H} = u\hat{x} - v\hat{y} \text{ around } \mathcal{C} = \operatorname{Re}\left(\oint_{\mathcal{C}} f(z)\,dz\right) \tag{1.32a}$$
$$\text{Flux of } \vec{H} = u\hat{x} - v\hat{y} \text{ through } \mathcal{C} = \operatorname{Im}\left(\oint_{\mathcal{C}} f(z)\,dz\right) \tag{1.32b}$$

This once again shows the convenience of working with the Pólya vector field $\vec{H} = u\hat{x} - v\hat{y}$; compare these statements to equations (8) and (9) in Zill and Wright, which have to make statements about the contour integral of the complex conjugate of $f(z)$:

$$\text{Circulation of } u\hat{x} + v\hat{y} \text{ around } \mathcal{C} = \operatorname{Re}\left(\oint_{\mathcal{C}} \overline{f(z)}\,dz\right) \tag{1.33a}$$
$$\text{Flux of } u\hat{x} + v\hat{y} \text{ through } \mathcal{C} = \operatorname{Im}\left(\oint_{\mathcal{C}} \overline{f(z)}\,dz\right) \tag{1.33b}$$

Practice Problems

18.1.1, 18.1.3, 18.1.5, 18.1.7, 18.1.13, 18.1.17, 18.1.19, 18.1.21, 18.1.23

Thursday 20 December 2012

2 The Cauchy-Goursat Theorem

On Tuesday, we learned how to integrate a function along a contour in the complex plane. If that function is analytic, as we will see, such integrals are much easier to evaluate. This is because of a key result called the Cauchy-Goursat theorem and its consequences. The two main results will be:

- For a closed curve $\mathcal{C}$, if a function $f(z)$ is analytic on $\mathcal{C}$ *and everywhere in the region bounded by* $\mathcal{C}$, the integral $\oint_{\mathcal{C}} f(z)\,dz$ of that function around that closed curve is zero.
- If two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ connect the same endpoints, and you can "deform" $\mathcal{C}_1$ into $\mathcal{C}_2$ without leaving the region in which $f(z)$ is analytic, the contour integrals $\int_{\mathcal{C}_1} f(z)\,dz$ and $\int_{\mathcal{C}_2} f(z)\,dz$ have the same value.

Let's see why these results are true, and how they can help us evaluate contour integrals.

2.1 Integral Around a Closed Loop

First consider the integral around a closed curve $\mathcal{C}$. Remember that we noted at the end of Tuesday's class that if $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, the contour integral of $f(z)$ around a closed curve is related to the flux and circulation of the Pólya vector field $\vec{H} = u\hat{x} - v\hat{y}$ around that curve:

$$\oint_{\mathcal{C}} f(z)\,dz = \oint_{\mathcal{C}} \vec{H}\cdot\hat{t}\,ds + i\oint_{\mathcal{C}} \vec{H}\cdot\hat{n}\,ds = \begin{pmatrix}\text{Circulation of } \vec{H}\\ \text{around } \mathcal{C}\end{pmatrix} + i\begin{pmatrix}\text{Flux of } \vec{H}\\ \text{through } \mathcal{C}\end{pmatrix} \tag{2.1}$$

There is a result from vector calculus known as Green's theorem, which has two equivalent statements, one relating the circulation of a vector field to its curl and the other relating the flux to the divergence. In both cases, let the closed curve $\mathcal{C}$ be the boundary of a region $\mathcal{R}$. Then Green's theorem in its two forms says

$$\oint_{\mathcal{C}} (F_x\,dx + F_y\,dy) = \oint_{\mathcal{C}} \vec{F}\cdot\hat{t}\,ds = \iint\limits_{\mathcal{R}} \left(\operatorname{curl}\vec{F}\right) d^2A = \iint\limits_{\mathcal{R}} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx\,dy \tag{2.2a}$$

$$\oint_{\mathcal{C}} (F_x\,dy - F_y\,dx) = \oint_{\mathcal{C}} \vec{F}\cdot\hat{n}\,ds = \iint\limits_{\mathcal{R}} \left(\operatorname{div}\vec{F}\right) d^2A = \iint\limits_{\mathcal{R}} \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) dx\,dy \tag{2.2b}$$

Applying this to the Pólya vector field in (2.1) gives us

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,dz &= \oint_{\mathcal{C}} [u\,dx - v\,dy] + i\oint_{\mathcal{C}} [u\,dy + v\,dx] = \oint_{\mathcal{C}} \vec{H}\cdot\hat{t}\,ds + i\oint_{\mathcal{C}} \vec{H}\cdot\hat{n}\,ds \\ &= \iint\limits_{\mathcal{R}} \left(\operatorname{curl}\vec{H}\right) d^2A + i\iint\limits_{\mathcal{R}} \left(\operatorname{div}\vec{H}\right) d^2A \\ &= \iint\limits_{\mathcal{R}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint\limits_{\mathcal{R}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy\end{aligned} \tag{2.3}$$

But recall that the divergence and curl of the Pólya vector field, which appear in the integrands of the integrals over $\mathcal{R}$, vanish when the Cauchy-Riemann equations are satisfied. So if $f(z)$ is analytic everywhere in $\mathcal{R}$, then the divergence and curl of $\vec{H} = u\hat{x} - v\hat{y}$ are zero everywhere in $\mathcal{R}$, which means the circulation of $\vec{H}$ around the boundary $\mathcal{C}$ and the flux of $\mathcal{H}$ through that boundary are zero, which means the integral of $f(z)$ around $\mathcal{C}$ is zero.

For an illustration of the relationship between flux and circulation and the integral of a function around a closed curve, see
http://demonstrations.wolfram.com/PolyaVectorFieldsAndComplexIntegrationAlongClosedCurves/

As specific example, consider $f(z) = z$; this is analytic everywhere. Let $\mathcal{C}$ be the circle of radius a centered at the origin; one parametrization is

$$z = ae^{it} \qquad t : 0 \to 2\pi \tag{2.4}$$

which has

$$dz = iae^{it}\,dt \tag{2.5}$$

We expect the integral around $\mathcal{C}$ to be zero. Let's check:

$$\oint_{\mathcal{C}} f(z)\,dz = \oint_{\mathcal{C}} z\,dz = \int_0^{2\pi} a\,e^{it}(iae^{it})dt = ia^2\int_0^{2\pi} e^{i2t}\,dt = \frac{a^2}{2}e^{i2t}\Big|_0^{2\pi} = \frac{a^2}{2}\left(e^{i4\pi} - e^{i0}\right) = 0 \tag{2.6}$$

As expected, the integral is zero.

Now, consider the function $f(z) = \frac{1}{z}$. It is analytic everywhere except the origin $z = 0$. Since the region bounded by $\mathcal{C}$, the unit disk, includes the origin, the Cauchy-Goursat theorem doesn't apply, and the integral need not be zero. Let's check:

$$\oint_{\mathcal{C}} f(z)\,dz = \oint_{\mathcal{C}} \frac{dz}{z} = \int_0^{2\pi} \frac{iae^{it}}{a\,e^{it}}dt = i\int_0^{2\pi} dt = 2\pi i \tag{2.7}$$

Notice two things:

- Even though $f(z)$ is analytic (not just differentiable) all along the curve $\mathcal{C}$, the Cauchy-Goursat theorem does not apply because $f(z)$ is not analytic at all points in the region $\mathcal{R}$ bounded by $\mathcal{C}$. And in fact $\oint_{\mathcal{C}} f(z)\,dz \neq 0$
- The integral $\oint_{\mathcal{C}} f(z)\,dz$ doesn't depend on the radius of the curve $\mathcal{C}$.

2.2 Independence of Path for Analytic Functions

Now to the second big consequence of the Cauchy-Goursat theorem, concerning the deformation of contours. Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two different curves that connect the same endpoints z_I and z_F. Then if we make the closed curve $\mathcal{C}$ by following $\mathcal{C}_1$ from z_I and z_F, then following $\mathcal{C}_2$ *backwards* from z_F back to z_I, we have

$$\oint_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}_1} f(z)\,dz - \int_{\mathcal{C}_2} f(z)\,dz \tag{2.8}$$

where the contribution from integrating backwards from z_F to z_I is minus the integral $\int_{\mathcal{C}_2} f(z)\,dz$ which results from integrating forwards along $\mathcal{C}_2$ from z_I to z_F. This means that **if $f(z)$ is analytic everywhere in the region between $\mathcal{C}_1$ and $\mathcal{C}_2$,** which is the region of which $\mathcal{C}$ is the boundary, then the integral around the closed loop is zero by the Cauchy-Goursat theorem, and

$$0 = \int_{\mathcal{C}_1} f(z)\,dz - \int_{\mathcal{C}_2} f(z)\,dz \tag{2.9}$$

i.e.

$$\int_{\mathcal{C}_1} f(z)\,dz = \int_{\mathcal{C}_2} f(z)\,dz \tag{2.10}$$

This is a very handy result. It means that **for analytic functions** we can "deform" a potentially inconvenient contour into one along which it's easier to calculate the integral.

Example: Evaluate

$$\int_{\mathcal{C}} \cos z\,dz \tag{2.11}$$

where $\mathcal{C}$ is the curve $y = \sin x$ from $z_I = 0$ to $z_F = \frac{\pi}{2} + i$.

Solution: Since $\cos z$ is analytic everywhere, we can evaluate the integral along any curve from $z_I = 0$ to $z_F = \frac{\pi}{2} + i$. In particular, consider the curve $\mathcal{C}_0$ which goes along the real axis from 0 to $\frac{\pi}{2}$, and then straight up in the imaginary direction from $\frac{\pi}{2}$ to $\frac{\pi}{2} + i$. The first piece is parametrized by x from 0 to $\frac{\pi}{2}$ with $z(x) = x$ and $dz = dx$; the second piece is parametrized by y from 0 to 1 with $z = \frac{\pi}{2} + iy$ and $dz = i\,dy$, so, recalling that $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

$$\begin{aligned}
\int_{\mathcal{C}} \cos z\,dz &= \int_{\mathcal{C}_0} \cos z\,dz \\
&= \int_0^{\pi/2} (\cos x \cosh 0 - i \sin x \sinh 0)\ dx + \int_0^1 \left(\cos\frac{\pi}{2}\cosh y - i \sin\frac{\pi}{2}\sinh y\right)(i\,dy) \\
&= \int_0^{\pi/2} \cos x\,dx + \int_0^1 \sinh y\,dy = \sin x\Big|_{x=0}^{x=\pi/2} + \cosh y\Big|_{y=0}^{y=1} \\
&= (1-0) + ([\cosh 1] - 1) = \cosh 1 = \frac{e - 4 + e^{-1}}{2} \approx 1.5431
\end{aligned} \tag{2.12}$$

2.3 Deformation of Closed Contours

We saw that the integral $\oint_{\mathcal{C}} f(z)\,dz$ was not necessarily zero, even if $f(z)$ was analytic all along $\mathcal{C}$, if there were one or more points in the region bounded by $\mathcal{C}$ where $f(z)$ was not analytic. We'll now show that in a case like that, you can still deform $\mathcal{C}$ without changing the value of the integral, i.e.,

$$\oint_{\mathcal{C}_1} f(z)\,dz = \oint_{\mathcal{C}_2} f(z)\,dz \tag{2.13}$$

as long as you can smoothly deform $\mathcal{C}_1$ into $\mathcal{C}_2$ while staying inside the region where $f(z)$ is analytic. The idea is to construct a closed contour which goes like this:

1. Counter-clockwise around $\mathcal{C}_1$ from some point P_1 back to that same point
2. From P_1 to a point P_2 on $\mathcal{C}_2$, along some curve $\mathcal{A}$
3. Clockwise (i.e., backwards) around $\mathcal{C}_2$ from P_2 back to P_2
4. From P_2 back to P_1, following $\mathcal{A}$ backwards (we can call this the curve $-\mathcal{A}$)

The integral around this curve $\mathcal{C}$ is the sum of the contributions from each piece:

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,dz &= \circlearrowleft\!\!\oint_{\mathcal{C}_1} f(z)\,dz + \int_{\mathcal{A}} f(z)\,dz + \circlearrowright\!\!\oint_{\mathcal{C}_2} f(z)\,dz + \int_{-\mathcal{A}} f(z)\,dz \\ &= \circlearrowleft\!\!\oint_{\mathcal{C}_1} f(z)\,dz + \int_{\mathcal{A}} f(z)\,dz - \circlearrowleft\!\!\oint_{\mathcal{C}_2} f(z)\,dz - \int_{\mathcal{A}} f(z)\,dz\end{aligned} \tag{2.14}$$

Since the two integrals along $\mathcal{A}$ in the opposite direction cancel out, we're left with

$$\circlearrowleft\!\!\oint_{\mathcal{C}} f(z)\,dz = \circlearrowleft\!\!\oint_{\mathcal{C}_1} f(z)\,dz - \circlearrowleft\!\!\oint_{\mathcal{C}_2} f(z)\,dz \tag{2.15}$$

If $f(z)$ is analytic in the region between $\mathcal{C}_1$ and $\mathcal{C}_2$, which is just the region inside the closed contour $\mathcal{C}$, then $\oint_{\mathcal{C}} f(z)\,dz$ is zero, and

$$\circlearrowleft\!\!\oint_{\mathcal{C}_1} f(z)\,dz = \circlearrowleft\!\!\oint_{\mathcal{C}_2} f(z)\,dz \tag{2.16}$$

So this idea of path independence works for closed curves as well.

The overall theme is that you can deform the contour around which a contour integral is evaluated, without changing the result of the integral, as long as you don't change the endpoints of the curve, or the overall direction for a closed curve, and you only move the curve through regions where the function is analytic.

2.4 The Antiderivative

In real single-variable calculus we can think of the derivative in two ways. The indefinite integral is the operation that undoes the derivative:

$$F(x) = f'(x) \Longleftrightarrow F(x) = \int f(x)\,dx \tag{2.17}$$

while the definite integral is the area under the function between two points x_1 and x_2:

$$\int_{x_1}^{x_2} f(x)\,dx = F(x_2) - F(x_1) \tag{2.18}$$

To evaluate the definite integral of $f(x)$, we only need to know the antiderivative $F(x)$ evaluated at the two endpoints.

For general complex functions, we only really have the equivalent of the definite integral. The contour integral

$$\int_{\mathcal{C}} f(z)\,dz \tag{2.19}$$

doesn't just depend on the endpoints of the curve $\mathcal{C}$, but it is a property of the curve itself. However, if a function is analytic in some domain (simply connected open set) $\mathcal{D}$, then we know that the contour integral of that function along any curve in $\mathcal{D}$ depends only on the endpoints of the curve. It is thus reasonable to write

$$\int_{z_I}^{z_F} f(z)\,dz = \int_{\mathcal{C}} f(z)\,dz \quad \mathcal{C} \text{ any curve in } \mathcal{D} \text{ from } z_I \text{ to } z_F;\ f(z) \text{ analytic in } \mathcal{D} \tag{2.20}$$

This then allows us to talk about an antiderivative $F(z)$ such that $f(z) = F'(z)$ everywhere in $\mathcal{D}$, and write

$$\int_{z_I}^{z_F} f(z)\,dz = F(z_I) - F(z_F) \tag{2.21}$$

For example, let $f(z) = \cos 2z$; then $F(z) = \frac{\sin 2z}{2}$, and

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}-i} \cos 2z\,dz = \frac{\sin(\pi - 2i)}{2} - \frac{\sin(\pi/2)}{2} \tag{2.22}$$

Since

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y \tag{2.23}$$

we know

$$\sin(\pi - 2i) = i(\cos\pi)\sinh(-2) = i(-1)(-\sinh 2) = i \sinh 2 \tag{2.24}$$

and of course $\sin(\pi/2) = 1$ so

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}-i} \cos 2z\,dz = \frac{i\sinh 2}{2} - \frac{1}{2} = -\frac{1}{2} + i\frac{e - e^{-1}}{4} \approx -0.5 + 1.8134i \tag{2.25}$$

Practice Problems

18.2.9, 18.2.11, 18.2.15, 18.2.21, 18.3.1, 18.3.7, 18.3.15, 18.3.17, 18.3.19

Tuesday 8 January 2013

Review for exam

Thursday 10 January 2013

First Prelim Exam

Tuesday 15 January 2013

3 Cauchy's Integral Formulas

We can use the Cauchy-Goursat theorem, and the ability to deform closed contours, to deduce some remarkable properties of analytic functions, which will form the basis for all of the considerations of complex power series to come in the following weeks. In particular, an analytic function will be seen to be infinitely differentiable, and the function and each of its derivatives at a point will determine the value of various integrals along contours around that point.

3.1 Cauchy's Integral Formula

To start with, consider the contour integral

$$\oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz \tag{3.1}$$

where $\mathcal{C}$ is any curve which goes counter-clockwise around z_0, and $f(z)$ is a function which is analytic in some domain $\mathcal{D}$ which completely contains the curve $\mathcal{C}$. Now, the function $\frac{f(z)}{z-z_0}$ is not necessarily analytic at z_0, but by the quotient rule, it's differentiable everywhere else that $f(z)$ is. That means that it's analytic on a region which consists of $\mathcal{D}$ with the point z_0 removed. So we don't change the value of the integral if we deform $\mathcal{C}$ into a circle $\mathcal{C}_\rho(z_0)$ of radius ρ centered on z_0, as long as ρ is small enough that $\mathcal{C}_\rho(z_0)$ lies completely within $\mathcal{D}$:

$$\oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz = \oint_{\mathcal{C}_\rho(z_0)} \frac{f(z)}{z - z_0}\, dz \tag{3.2}$$

Now, we can make ρ as small as we like (as long as it remains positive, so to as good an approximation as we like, we can replace the $f(z)$ in the integrand with the constant $f(z_0)$:

$$\frac{f(z)}{z - z_0} \approx \frac{f(z_0)}{z - z_0} \qquad \Longrightarrow \qquad \oint_{\mathcal{C}_\rho(z_0)} \frac{f(z)}{z - z_0}\, dz = \oint_{\mathcal{C}_\rho(z_0)} \frac{f(z_0)}{z - z_0}\, dz \tag{3.3}$$

and then, since $f(z_0)$ is a constant, we can take it outside the integral, and get

$$\oint_{\mathcal{C}_\rho(z_0)} \frac{f(z_0)}{z - z_0} = f(z_0) \oint_{\mathcal{C}_\rho(z_0)} \frac{dz}{z - z_0} \tag{3.4}$$

We can evaluate the contour integral using the parametrization $z = z_0 + \rho e^{it}$, $dz = i\rho e^{it}\, dt$ and find

$$\oint_{\mathcal{C}_\rho(z_0)} \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i\rho e^{it}\, dt}{\rho e^{it}} = 2\pi i \tag{3.5}$$

which means that

$$\oint_{\mathcal{C}} \frac{f(z)}{z-z_0}\,dz = \oint_{\mathcal{C}_\rho(z_0)} \frac{f(z_0)}{z-z_0}\,dz = 2\pi i f(z_0) \tag{3.6}$$

Strictly speaking, to show this we need to show that

$$\oint_{\mathcal{C}_\rho(z_0)} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0 \tag{3.7}$$

which is done formally in Zill and Wright by showing that

$$\left|\oint_{\mathcal{C}_\rho(z_0)} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \tag{3.8}$$

is smaller than any positive number you choose, using the ML limit.

3.1.1 Example #1

Example: integrate

$$\oint_{\mathcal{C}} \frac{e^z}{z+i}\,dz \tag{3.9}$$

where $\mathcal{C}$ is a counter-clockwise circle of radius 2 centered on the origin.

We can write the integrand in the form

$$\frac{e^z}{z+i} = \frac{f(z)}{z-z_0} \tag{3.10}$$

where $f(z) = e^z$ and $z_0 = -i$. Our $f(z)$ is analytic everywhere, and our $z_0 = -i$ is inside the contour described, a circle of radius 2 centered on the origin. We can thus use Cauchy's integral formula to write

$$\oint_{\mathcal{C}} \frac{e^z}{z+i}\,dz = \oint_{\mathcal{C}} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) = 2\pi i e^{-i} = 2\pi i(\cos 1 - i\sin 1) = 2\pi(\sin 1 + i\cos 1) \tag{3.11}$$

3.1.2 Example #2

Example: integrate

$$\oint_{\mathcal{C}} \frac{1}{2z^2-7iz-3}\,dz \tag{3.12}$$

where $\mathcal{C}$ is a counter-clockwise circle of radius 1 centered on the origin.

This one is a little trickier, since the numerator is not already in the form $z - z_0$. But we can do it if note that we can factor the denominator into

$$2z^2 - 7iz - 3 = (2z-i)(z-3i) = 2\left(z-\frac{i}{2}\right)(z-3i) \tag{3.13}$$

Now, this goes to zero at $z = \frac{i}{2}$ and also at $z = 3i$. Note that the first point is inside the contour $\mathcal{C}$, and the second is outside. So we can write

$$\frac{1}{2z^2 - 7iz - 3} = \frac{\frac{1}{2(z-3i)}}{z - \frac{i}{2}} = \frac{f(z)}{z - z_0} \tag{3.14}$$

with

$$f(z) = \frac{1}{2(z - 3i)} \tag{3.15}$$

and

$$z = \frac{i}{2} \tag{3.16}$$

Note that it was really important to factor the integrand so that the denominator was $z - \frac{i}{2}$ and not $2z - i$, in order to have the form $\frac{f(z)}{z-z_0}$.

Now $f(z)$ is not analytic everywhere (it blows up at $z = 3i$), but it is analytic everywhere on and inside the circle $\mathcal{C}$, so it satisfies the conditions for Cauchy's integral formula. We can thus write

$$\oint_{\mathcal{C}} \frac{1}{2z^2 - 7iz - 3}\, dz = \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) = 2\pi i \frac{1}{2(i/2 - 3i)} = 2\pi i \frac{1}{i - 6i} = -\frac{2\pi}{5} \tag{3.17}$$

3.2 Cauchy's Integral Formula for Derivatives

The formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz \tag{3.18}$$

is the $n = 0$ case of a general formula for the nth derivative of $f^{(n)}(z)$ evaluted at $z = z_0$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad \text{valid for any non-negative integer } n \tag{3.19}$$

Note that if we knew $f(z)$ could be described by a Taylor series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \tag{3.20}$$

then it would be trivial to show

$$\frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{n!}{2\pi i} \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \oint_{\mathcal{C}} (z - z_0)^{k-n-1}\, dz = f^{(k)}(z_0) \tag{3.21}$$

because it's easy to show by deformation of contours that

$$\oint_{\mathcal{C}} (z - z_0)^{k-n-1}\, dz = \begin{cases} 0 & k \neq n \\ 2\pi i & k = n \end{cases} \tag{3.22}$$

However, this would be completely cheating, because the formula (3.18) will be used to *show* that an analytic function is infinitely differentiable, and also that is equal to its Taylor series.

So instead, to avoid circular logic, there's a proof by induction, where the integral formula for $f'(z_0)$ can be derived using the formula for $f(z_0)$, then the formula for $f''(z_0)$ can be derived using the formula for $f'(z_0)$, etc. The fundamentals of the method go like this: if we've already shown that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \tag{3.23}$$

for some n, we can write

$$\begin{aligned} f^{(n+1)}(z_0) &= \lim_{\Delta z\to 0} \frac{f^{(n)}(z_0+\Delta z) - f^{(n)}(z_0)}{\Delta z} \\ &= \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left(\frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-z_0-\Delta z)^{n+1}}\,dz - \frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right) \\ &= \lim_{\Delta z\to 0}\left(\frac{n!}{2\pi i}\oint_{\mathcal{C}} f(z)\frac{(z-z_0-\Delta z)^{-(n+1)} - (z-z_0)^{-(n+1)}}{\Delta z}\,dz\right) \end{aligned} \tag{3.24}$$

where $\mathcal{C}$ is some curve that lies within the domain where $f(z)$ is analytic, and goes counterclockwise around both z_0 and $z_0 + \Delta z$. We can show, for example using l'Hôpital's rule, that

$$\lim_{\Delta z\to 0} \frac{(z-z_0-\Delta z)^{-(n+1)} - (z-z_0)^{-(n+1)}}{\Delta z} = (n+1)(z-z_0)^{-(n+2)} \tag{3.25}$$

and so

$$f^{(n+1)}(z_0) = \frac{n!(n+1)}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+2}}\,dz = \frac{(n+1)!}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+2}}\,dz \tag{3.26}$$

which is indeed the Cauchy integral formula for $f^{(n+1)}(z_0)$.

Example: Zill & Wright problem 18.4.23.

As noted a moment ago, Cauchy's integral formula for derivatives shows that if a function $f(z)$ is analytic at a point, we can construct any derivative we like by defining a contour integral on a contour which winds counterclockwise around that point. This means that if a function is analytic (which required that it be differentiable in some neighborhood containing that point, and that the relevant first derivatives be continous in that neighborhood), it's actually infinitely differentiable. This is definitely not the case for real functions. Consider for example the function

$$f(x) = \begin{cases} \frac{x^2}{2} & x \le 0 \\ 1-\cos x & x \ge 0 \end{cases} \tag{3.27}$$

For this function $f(x)$ and $f'(x)$ are continuous at the origin, as are $f''(x)$ and $f^{(3)}(x)$, but $f^{(4)}(x)$ is discontinuous at the origin, and $f^{(4)}(0)$ is not defined.

3.3 Consequences of Cauchy's Integral Formulas

3.3.1 Cauchy's Inequality

If an analytic function is bounded along a circle $\mathcal{C}_\rho(z_0)$ of radius ρ of radius ρ centered on a point z_0

$$|f(z)| \leq M \qquad \text{when } |z - z_0| = \rho \tag{3.28}$$

we can use Cauchy's Integral Formulas together with the ML bound to set a bound on the derivatives of $f(z)$ at any point z_0 inside $\mathcal{C}$. Specifically,

$$|f^n(z_0)| = \left| = \frac{n!}{2\pi i} \oint_{\mathcal{C}_\rho(z_0)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \right| \leq \frac{n!}{2\pi} \frac{M}{\rho^{n+1}} (2\pi\rho) = \frac{n! M}{\rho^n} \tag{3.29}$$

3.3.2 Liouville's Theorem

Cauchy's inequality seems sort of academic, but it can be used to demonstrate something more surprising: any function which is analytic everywhere (which we call an entire function), and also bounded everywhere, is a constant. We can do this by using Cauchy's inequality with $n = 1$ to show the derivative at a point z_0 obeys the inequality

$$|f'(z_0)| \leq \frac{\max_{\mathcal{C}_\rho(z_0)} |f(z)|}{\rho} \leq \frac{\max |f(z)|}{\rho} \tag{3.30}$$

but if the function is analytic everywhere, we can make ρ arbitrarily large, and the right-hand side arbitrarily small. That means $|f'(z_0)|$ must be zero, which means $f'(z_0)$ must be zero. But we can do this at any point, so $f'(z) = 0$ everywhere and therefore $f(z)$ is a constant.

Note that no such restriction exists for real functions: $f(x) = \frac{1}{1+x^2}$ is infinitely differentiable for every real x, and obeys $|f(x)| \leq 1$ for every real x.

Practice Problems

18.4.1, 18.4.3, 18.4.7, 18.4.9, 18.4.15, 18.4.17, 18.4.19, 18.4.21, 18.4.23