Series and Residues
(Zill & Wright Chapter 19)

1016-420-02: Complex Variables*

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Contents

Thursday 17 January 2013

1 Sequences and Series in the Complex Plane

1.1 Motivation and Perspective

Sequences and series can be sort of an abstract subject in calculus, whether real or complex. It's useful to keep an eye on why we're considering all these things, so let's take a top-down rather than bottom-up approach. We're fundamentally interested in a power series, which is a complex function of z of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{1.1}$$

where the $\{a_k\}$, as well as z_0, are complex constants. This is a special case of a general infinite series

$$\sum_{k=0}^{\infty} w_k \tag{1.2}$$

where $\{w_k\}$ is a progression of complex numbers. The infinite series is implicitly defined as a limit

$$\sum_{k=0}^{\infty} w_k = \lim_{n\to\infty} \sum_{k=0}^{n} w_k = \lim_{n\to\infty} S_n \tag{1.3}$$

which is the limit of a sequence of partial sums $\{S_n\}$. If that limit exists, we say the sequence $\{S_n\}$ *converges*. Formally, the limit

$$L = \lim_{n\to\infty} S_n \tag{1.4}$$

means that for any $\epsilon > 0$ there is an N such that $|S_n - L| < \epsilon$ whenever $n > N$. It's not too hard to see that

$$L = \lim_{n\to\infty} S_n \qquad \text{if and only if} \qquad \mathrm{Re}(L) = \lim_{n\to\infty} \mathrm{Re}(S_n) \text{ and } \mathrm{Im}(L) = \lim_{n\to\infty} \mathrm{Im}(S_n) \tag{1.5}$$

1.2 Tests for Convergence

You may recall from real calculus an array of tests for whether an infinite series converged. There are corresponding tests in complex analysis, and we quote the most important of them.

1.2.1 The nth Term Test

A nearly trivial test is whether the terms in a series are getting progressively smaller with increasing n:

$$\sum_{k=0}^{\infty} w_k \text{ converges } \textbf{only} \text{ if } \lim_{n\to\infty} w_n = 0 \tag{1.6}$$

This is not so useful, though, since it can only be used to show that a series diverges, not that it converges.

1.2.2 Absolute Convergence and the Ratio Test

The concept of absolute convergence exists for complex series as well as real ones, but now the absolute value in question is the modulus:

$$\sum_{k=0}^{\infty} w_k \text{ converges absolutely} \qquad \text{means} \qquad \sum_{k=0}^{\infty} |w_k| \text{ converges} \tag{1.7}$$

Absolute convergence is used in the *ratio test*, which examines the limit of $\left|\frac{w_{k+1}}{w_k}\right|$:

- If $\lim_{k\to\infty} \left|\frac{w_{k+1}}{w_k}\right| = L < 1$, $\sum_{k=0}^{\infty} w_k$ converges absolutely.
- If $\lim_{k\to\infty} \left|\frac{w_{k+1}}{w_k}\right| = L > 1$, or $\lim_{k\to\infty} \left|\frac{w_{k+1}}{w_k}\right| = \infty$, $\sum_{k=0}^{\infty} w_k$ diverges.
- If $\lim_{k\to\infty} \left|\frac{w_{k+1}}{w_k}\right| = 1$, we can't say whether the series converges or not.

1.3 Radius of Convergence

We now want to consider the question of convergence for a power series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{1.8}$$

in which $w_k = a_k(z - z_0)^k$. Since the terms in the series depend on z, the series will in general converge for some values of z and diverge for some others. If we apply the ratio test, the quantity in question is

$$\lim_{k\to\infty} \left|\frac{w_{k+1}}{w_k}\right| = \lim_{k\to\infty} \left|\frac{a_{k+1}(z-z_0)^{k+1}}{a_k(z-z_0)^k}\right| = |z - z_0| \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \tag{1.9}$$

So,

- If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 0$, then the series converges absolutely, no matter what z is.
- If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \infty$, then the series will only converge if $z = z_0$. (If $z = z_0$, then the series only has one term, a_0.)

- If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{R}$ where $0 < R < \infty$, then the series will converge absolutely if

$$\frac{|z-z_0|}{R} < 1 \tag{1.10}$$

i.e., for any z such that

$$|z-z_0| < R\ . \tag{1.11}$$

It will diverge if

$$\frac{|z-z_0|}{R} > 1 \tag{1.12}$$

i.e., for any z such that

$$|z-z_0| > R \tag{1.13}$$

Since $|z-z_0| = R$, which separates the regions where the power series converges and diverges, is a circle of radius R centered on z_0, R is known as the *radius of convergence* for the power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$. The series converges absolutely everywhere inside this circle of convergence, and diverges outside it. For points on the circle, it may converge or diverge.

If a power series converges for all z, we say its radius of convergence is infinite; if it converges only when $z = z_0$, its radius of convergence is said to be zero.

1.4 Geometric Series

Just now we saw that a power series $\sum_{k=0}^{\infty} a_k(z-z_0)$ converges inside some circle centered at z_0, whose radius R is called the *radius of convergence* for the series. I.e.,

$$\sum_{k=0}^{\infty} a_k(z-z_0) \text{ converges when } |z-z_0| < R \text{ and diverges when } |z-z_0| > R \tag{1.14}$$

For another example of this, consider the geometric series

$$1 + z + z^2 + \ldots = \sum_{k=0}^{\infty} z^k \tag{1.15}$$

If we multiply the partial sum $S_n = \sum_{k=0}^{n} z^k$ by $1-z$, we find

$$(1-z)S_n = (1-z)\sum_{k}^{n} z^k = (1-z)(1+z+z^2+\ldots+z^n) = 1-z^{n+1} \tag{1.16}$$

but that means

$$S_n = \frac{1-z^{n+1}}{1-z} = \frac{1}{1-z} - \frac{z^{n+1}}{1-z} \tag{1.17}$$

The second term goes to zero as n goes to infinity, if $|z| < 1$, and it diverges if $|z| > 1$. So

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \qquad \text{if } |z| < 1 \tag{1.18}$$

The series $\sum_{k=0}^{\infty} z^k$ is called a geometric series, and its circle of convergence is a circle of radius 1 centered on the origin.

This also works with other expressions substituted for z, so

$$1 - z + z^2 - z^3 + \dots \sum_{k=0}^{\infty} (-z)^k = \frac{1}{1+z} \qquad \text{if } |z| < 1 \tag{1.19}$$

and

$$\sum_{k=0}^{\infty} (z-i)^k = \frac{1}{1+i-z} \qquad \text{if } |z-i| < 1 \tag{1.20}$$

etc.

Practice Problems

19.1.3, 19.1.7, 19.1.9, 19.1.11, 19.1.15, 19.1.19, 19.1.21, 19.1.23, 19.1.25, 19.1.27

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2 Taylor Series

A Taylor series is a power series of the form

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \tag{2.1}$$

which, for some neighborhood of the point z_0, converges to a function $f(z)$ which is analytic in the same neighborhood. There are various theorems stated in Zill and Wright which say that you can do sensible things with a convergent series, like differentiate and integrate it "term-by-term"; given that you can do these things, you can work out the coëfficients $\{a_k\}$ by requiring the function and all of its derivatives to match up with the series at the point $z = z_0$:

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \tag{2.2a}$$
$$f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \dots \tag{2.2b}$$
$$f''(z) = 2a_2 + 3 \cdot 2a_3(z-z_0) + 4 \cdot 3a_4(z-z_0)^2 + \dots \tag{2.2c}$$
$$f^{(3)}(z) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(z-z_0) + 5 \cdot 4 \cdot 3a_5(z-z_0)^2 + \dots \tag{2.2d}$$
$$\vdots \tag{2.2e}$$

so

$$f(z_0) = a_0 \tag{2.3a}$$
$$f'(z_0) = a_1 \tag{2.3b}$$
$$f''(z_0) = 2a_2 \tag{2.3c}$$
$$f^{(3)}(z_0) = 3 \cdot 2a_3 \tag{2.3d}$$
$$\vdots \tag{2.3e}$$

It's not so hard to see the general pattern

$$f^{(n)}(z_0) = n!a_n \tag{2.4}$$

so the coëfficients are

$$a_n = \frac{f^{(n)}(z_0)}{n!} \tag{2.5}$$

and

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \qquad \text{when the series converges} \tag{2.6}$$

This is called the Taylor series for $f(z)$ with center z_0. As a special case, the Taylor series with center 0 is called the Maclaurin series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \qquad \text{when the series converges} \tag{2.7}$$

The question we haven't addressed yet is for which z the Taylor series converges. Since it's a power series, we expect it to converge inside some circle centered on z_0. Moving out from z_0, the series ought to be well-behaved (i.e., convergent) as long as the function is well-behaved (i.e., analytic). So it shouldn't be surprising that the radius of convergence is the distance from z_0 to the nearest point where $f(z)$ fails to be analytic.

You can show this using Cauchy's integral theorems. If we can draw a circle $\mathcal{C}_R(z_0)$ centered on z_0, of radius $R > |z - z_0|$, which lies entirely in the domain where the function is analytic, we can use Cauchy's integral theorem to write

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{s - z} \, ds \tag{2.8}$$

where we call the integration variable s, since it's a letter we haven't used before. Note that $\mathcal{C}_R(z_0)$ is not centered on z, but z lies inside the curve, which is all we need for Cauchy's integral theorem. Since we're interested in derivatives at z_0, we write the denominator as

$$s - z = s - z_0 - (z - z_0) = (s - z_0)\left(1 - \frac{z - z_0}{s - z_0}\right) \tag{2.9}$$

We can then use the geometric series, with $\frac{z-z_0}{s-z_0}$ in place of z, to write

$$\frac{1}{s-z} = \frac{1}{s-z_0}\left(1 + \frac{z-z_0}{s-z_0} + \frac{(z-z_0)^2}{(s-z_0)^2} + \ldots + \frac{(z-z_0)^n}{(s-z_0)^n} + T_{n+1}\right) \tag{2.10}$$

where the remainder term is

$$T_{n+1} = \frac{\frac{(z-z_0)^{n+1}}{(s-z_0)^{n+1}}}{1-\frac{z-z_0}{s-z_0}} = \frac{(z-z_0)^{n+1}}{(s-z_0)^{n+1}}\frac{s-z_0}{s-z_0-(z-z_0)} = \frac{(z-z_0)^{n+1}}{(s-z_0)^{n+1}}\frac{s-z_0}{s-z} \tag{2.11}$$

Since s is the integration variable along the circle of radius R centered at z_0, $|s-z_0| = R$, and since z is a point inside that circle,

$$|z-z_0| < R = |s-z_0| \tag{2.12}$$

which means that

$$\left|\frac{z-z_0}{s-z_0}\right| < 1 \tag{2.13}$$

which means that we should not be surprised that the remainder term goes to zero as $n \to \infty$, although the real demonstration considers this a little more carefully. Keeping that term for now, we have

$$\frac{1}{s-z} = \frac{1}{s-z_0} + \frac{z-z_0}{(s-z_0)^2} + \frac{(z-z_0)^2}{(s-z_0)^3} + \ldots + \frac{(z-z_0)^n}{(s-z_0)^{n+1}} + \frac{(z-z_0)^{n+1}}{(s-z)(s-z_0)^{n+1}}\ ; \tag{2.14}$$

when we substitute that into (2.8) it gives us

$$\begin{aligned} f(z) = {} & \frac{1}{2\pi i}\oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{s-z_0}\,ds + \frac{z-z_0}{2\pi i}\oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{(s-z_0)^2}\,ds + \ldots \\ & + \frac{(z-z_0)^n}{2\pi i}\oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{(s-z_0)^{n+1}}\,ds + \frac{(z-z_0)^{n+1}}{2\pi i}\oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{(s-z)(s-z_0)^{n+1}}\,ds \end{aligned} \tag{2.15}$$

We can apply Cauchy's integral theorem for derivatives, using the point z_0 rather than z this time, to each of the integrals except the last one and get

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \ldots + \frac{f^n(z_0)}{n!}(z-z_0)^n + R_{n+1}(z) \tag{2.16}$$

where the remainder integral is

$$R_{n+1}(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{(s-z)(s-z_0)^{n+1}}\,ds \tag{2.17}$$

which we can show goes to zero using the ML limit. If we write $|z-z_0| = \rho$, we've already noted that $\rho < R = |s-z_0|$. If we think about the geometry in the complex plane, z_0, z, and s are the corners of a triangle, and the triangle inequality says that

$$|s-z| \geq |s-z_0| - |z-z_0| = R - \rho \tag{2.18}$$

so

$$\frac{1}{|s-z|} \leq \frac{1}{R-\rho} \tag{2.19}$$

Since $f(s)$ is analytic along $\mathcal{C}_R(z_0)$, it must remain finite all along that circle, and so we can write the maximum of is modulus as M and then $|f(s)| \leq M$ all along $\mathcal{C}_R(z_0)$. Thus the modulus of the integrand is bounded by

$$\left|\frac{f(s)}{(s-z)(s-z_0)^{n+1}}\right| \leq \frac{M}{(R-\rho)R^{n+1}} \; ; \tag{2.20}$$

the length of the curve is the circumference $2\pi R$, so

$$|R_{n+1}(z)| = \frac{\rho}{2\pi}\left|\oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{(s-z)(s-z_0)^{n+1}}\,ds\right| \leq \frac{\rho^{n+1}}{2\pi}\frac{M}{(R-\rho)R^{n+1}}(2\pi R) = \frac{MR}{R-\rho}\left(\frac{\rho}{R}\right)^{n+1} \tag{2.21}$$

which does indeed go to zero as $n \to \infty$, because

$$|z-z_0| = \rho < R = |s-z_0| \tag{2.22}$$

so we have, for any z inside a circle centered on z_0 which fits inside the domain of analyticity of $f(z)$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \qquad |z-z_0| < \text{radius of convergence} \tag{2.23}$$

2.1 Calculation of Taylor Series

In practice, we only actually use the explicit form of the Taylor series coëfficients in terms of the derivatives for a few functions. Mostly we use the Maclaurin series for e^z (and the related series for $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$) and for $\frac{1}{1-z}$ with appropriate expressions in place of z.

For example, if $f(z) = e^z$, then $f^{(k)}(z) = e^z$, so $f^{(k)}(0) = 1$ and

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{2.24}$$

from which we can find, e.g.,

$$\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{z^k + (-z)^k}{2} = \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{z^k}{k!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \tag{2.25}$$

We can expand the same function in Taylor series around different origins. Consider the function $f(z) = \frac{1}{z+i}$; we know this is analytic everywhere except at $z = -i$. Let's expand it first in a Maclaurin series, and then in a Taylor series about the center $z = 1$.

To expand in a Maclaurin series, write

$$\frac{1}{z+i} = \frac{-i}{1-iz} = -i\sum_{k=0}^{\infty}(iz)^k = \sum_{k=0}^{\infty} i^{k-1}z^k \tag{2.26}$$

This series has a radius of convergence equal to the distance from $z=0$ to $z=-i$, which is $R=1$.

On the other hand, if we want to use the center $z_0=1$, we need to expand in powers of $z-1$. To do this we write

$$\frac{1}{z+i} = \frac{1}{1+i+(z-1)} \tag{2.27}$$

Next we want to divide the numerator and denominator by $1+i$, noting that

$$\frac{1}{1+i} = \frac{1}{1+i}\frac{1-i}{1-i} = \frac{1-i}{2} \tag{2.28}$$

so

$$\frac{1}{z+i} = \frac{1-i}{2}\frac{1}{1-\frac{-1+i}{2}(z-1)} = \frac{1-i}{2}\sum_{k=0}^{\infty}\left(\frac{-1+i}{2}\right)^k (z-1)^k = \sum_{k=0}^{\infty}(-1)\left(\frac{-1+i}{2}\right)^{k+1}(z-1)^k \tag{2.29}$$

This series has a radius of convergence equal to the distance from the center $z_0=1$ to the singularity $-i$, which is $\sqrt{2}$.

2.2 Taylor Series and Branch Cuts

Exercise 19.2.31

Practice Problems

19.2.1, 19.2.3, 19.2.9, 19.2.11, 19.2.13, 19.2.19, 19.2.23, 19.2.27, 19.2.29, 19.2.35

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3 Laurent Series

The Taylor series is a way of writing an analytic function $f(z)$ as a series

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \tag{3.1}$$

inside some disk $|z-z_0|<R$ in which $f(z)$ is analytic. The Laurent series is an extension to this which is valid in some annulus (ring) $r<|z-z_0|<R$. In general it includes negative

powers of $z - z_0$, so it looks like

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (3.2)$$

The negative powers of $z - z_0$ means it won't converge at $z = z_0$, and there are generally one or more points in the circle $|z - z_0| \leq r$ for which the function is not analytic.

Consider, for example, the function $f(z) = \frac{\sin z}{z^3}$. This is analytic everywhere except at $z = 0$, but since it is undefined there, we cannot expand it in a Taylor series about $z = 0$. But we know that $\sin z$ has the Taylor series

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{(-1)^{(k-1)/2}}{k!} z^k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad (3.3)$$

so it follows that

$$f(z) = \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \ldots = \sum_{\substack{k=-2 \\ k \text{ even}}}^{\infty} \frac{(-1)^{(k+2)/2}}{(3+k)!} z^k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-2} \quad (3.4)$$

This series converges for all z except $z = 0$, i.e., for $0 < |z|$.

Just as Taylor's theorem shows that there always exists a Taylor series which converges in a circular region in which $f(z)$ is analytic, by using Cauchy's integral formula on a inside that region which encloses both z and z_0, so there is a Laurent's theorem which shows that there is always a Laurent series which converges in an annular region in which $f(z)$ is analytic. Remember that by using Cauchy's integral formula and then expanding

$$\frac{1}{s - z} = \frac{1}{(s - z_0) - (z - z_0)} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(s - z_0)^{k+1}} \quad (3.5)$$

valid when $|s - z_0| > |z - z_0|$, we could write

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{s - z}\, ds = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{2\pi i} \oint_{\mathcal{C}_R(z_0)} \frac{f(s)}{(s - z_0)^{k+1}}\, ds \quad (3.6)$$

We then used Cauchy's integral theorem for derivatives to write the second part in terms of derivatives of the function $f(z)$, evaluated at z_0, but that second step won't be applicable for the Laurent series demonstration, since $f(z)$ is not analytic everywhere inside the curve, just in the annulus.

The starting point is also somewhat different, since it requires a closed curve which encloses z and lies within the annulus $r < |z - z_0| < R$. Since the function $f(z)$ needs to be analytic everywhere inside the curve, it can't actually have any points where $|z - z_0| < r$ inside it. So we need to use some C-shaped curve $\mathcal{K}$ made up of four parts:

1. A counter-clockwise circle $\mathcal{C}_{R_2}(z_0)$ once around the annulus at radius R_2 where $|z-z_0| < R_2 < R$

2. An inward cut from the radius R_2 to r_1 where $r < r_1 < |z-z_0|$

3. A clockwise circle once around the annulus at radius r_1

4. An outward cut from the radius r_1 back to R_2

The second and fourth pieces cancel out, and the first an third pieces mean

$$f(z) = \frac{1}{2\pi i}\oint_{\mathcal{K}} \frac{f(s)}{s-z}\,ds = \frac{1}{2\pi i}\oint_{\mathcal{C}_{R_2}(z_0)} \frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\oint_{\mathcal{C}_{r_1}(z_0)} \frac{f(s)}{s-z}\,ds \tag{3.7}$$

In both integrals, we write

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} \tag{3.8}$$

By construction, the first integral has $|s-z_0| = R_2 > |z-z_0|$ and the second integral has $|s-z_0| = r_1 < |z-z_0|$. In the first integral we can expand, as in the case of the Taylor series,

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = \frac{1}{s-z_0}\frac{1}{1-\frac{z-z_0}{s-z_0}} = \sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(s-z_0)^{k+1}} \tag{3.9}$$

which converges because $\left|\frac{z-z_0}{s-z_0}\right| < 1$. On the other hand, in the second series, we can write

$$\frac{1}{s-z} = \frac{1}{(s-z_0)-(z-z_0)} = -\frac{1}{z-z_0}\frac{1}{1-\frac{s-z_0}{z-z_0}} = -\sum_{k=0}^{\infty}\frac{(s-z_0)^k}{(z-z_0)^{k+1}} \tag{3.10}$$

This allows us to write

$$f(z) = \sum_{k=0}^{\infty}\frac{(z-z_0)^k}{2\pi i}\oint_{\mathcal{C}_{R_2}(z_0)}\frac{f(s)}{(s-z_0)^{k+1}},ds + \sum_{k=0}^{\infty}\frac{(z-z_0)^{-k-1}}{2\pi i}\oint_{\mathcal{C}_{r_1}(z_0)}\frac{f(s)}{(s-z_0)^{-k}},ds\ . \tag{3.11}$$

(There is a more careful statement in terms of remainder integrals and the ML-limit in Zill and Wright.) If we reorganize the summation index in the second series as $n = -k-1$, and rename the summation index in the first series we get

$$f(z) = \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{2\pi i}\oint_{\mathcal{C}_{R_2}(z_0)}\frac{f(s)}{(s-z_0)^{n+1}},ds + \sum_{n=-\infty}^{-1}\frac{(z-z_0)^n}{2\pi i}\oint_{\mathcal{C}_{r_1}(z_0)}\frac{f(s)}{(s-z_0)^{n+1}},ds \tag{3.12}$$

Since the integrand $\frac{f(s)}{(s-z_0)^{n+1}}$ is analytic everywhere in the annulus $r < |s-z_0| < R$, we can replace the contours in both integrals with any contour $\mathcal{C}$ which goes once clockwise around the annulus, and combine the sums:

$$f(z) = \sum_{n=-\infty}^{\infty}\frac{(z-z_0)^n}{2\pi i}\oint_{\mathcal{C}}\frac{f(s)}{(s-z_0)^{n+1}},ds \tag{3.13}$$

In practice we never find the coëfficients by evaluating these contour integrals; we either start from a Taylor series or we apply a trick using the geometric series

$$\frac{1}{1-w} = \sum_{k=0}^{\infty} w^k \qquad |w| < 1 \tag{3.14}$$

3.1 Calculation of Laurent Series

3.1.1 Example 19.3.17

3.1.2 Example 19.3.19

Practice Problems

19.3.1, 19.3.3, 19.3.7, 19.3.9, 19.3.11, 19.3.13, 19.3.15, 19.3.21, 19.3.25, 19.3.27

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4 Zeros, Poles and Residues

4.1 Poles and Zeros

Last week we saw that a function which was analytic in some annular region $r_1 < |z - z_0| < r_2$ could be expanded in a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{4.1}$$

which converged in that region. Typically, we get different series in concentric annuli centered on z_0, which have singular points of $f(z)$ at their borders. Consider the innermost region $0 < |z - z_0| < R$, and split the Laurent series valid in that region into negative and non-negative powers of $z - z_0$:

$$f(z) = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k + \sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{4.2}$$

The sum of positive powers $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ is called the *analytic part* of the series (since it converges to an analytic function) and the sum of negative powers $\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k}$ is called the *principal part*. It is useful to categorize a series by the most negative power of $z - z_0$ in its Laurent series:

- If the principal part of the series vanishes, but the original function $f(z)$ is not continuous at $z = z_0$, we say the function has a *removable singularity* at $z = z_0$. For example, consider

$$f(z) = \frac{\sin z}{z} \tag{4.3}$$

which us undefined at $z = 0$. Since the Maclaurin series for $\sin z$ is

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \ , \tag{4.4}$$

the Laurent series for $f(z)$, valid for $0 < |z|$, is

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \tag{4.5}$$

which converges to 1 at $z = 0$. Of course, in this case, we can define a function

$$g(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases} \tag{4.6}$$

which "removes" the singularity.

- If a_{-n} is non-zero but the principal part of the series has no terms a_k with $k < -n$, i.e.,

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \dots \tag{4.7}$$

then we say that $f(z)$ has a pole of order n at z_0. A pole of order 1 is also called a *simple pole*. For example, since

$$\frac{z+i}{(z-i)^2} = \frac{z - i + 2i}{(z-i)^2} = \frac{2i}{(z-i)^2} + \frac{1}{z-i} \tag{4.8}$$

we say that $\frac{z+i}{(z-i)^2}$ has a pole of order 2 at $z = i$.

- If the principal part of the Laurent series contains arbitrarily negative powers of $z - z_0$, then the function has an *essential singularity* at $z = z_0$. For example,

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{z}\right)^k = \sum_{k=0}^{\infty} \frac{z^{-k}}{k!} \tag{4.9}$$

has an essential singularity at $z = 0$.

It is very important to keep in mind that the Laurent series to use is the one valid in a "punctured neighborhood" of the point $z = z_0$. See Chapter review question 19.3.

Note that if the lowest power appearing in the Laurent series is positive, then the series converges to a regular function which is zero at z_0. If the first non-zero term in the sum is $a_n(z - z_0)^n$, with $n > 0$, we say the function has a *zero of order* n at $z = z_0$.

Often the way to figure out the order of a pole is to write the function as a ratio of analytic functions. If $f(z_0) \neq 0$, and $g(z)$ has a zero of order n at z_0, then $f(z)/g(z)$ has a pole of order n at z_0. So for example,

$$\frac{(z+2)e^z}{(z-3)^2(z+i)} \tag{4.10}$$

has a pole of order 2 at $z = 3$ and a pole of order 1 at $z = i$. However,

$$\frac{z^2 - 1}{(z-1)^2} \tag{4.11}$$

has a pole at $z = 1$ of order one, not two, because

$$\frac{z^2 - 1}{(z-1)^2} = \frac{(z+1)(z-1)}{(z-1)^2} = \frac{z+1}{z-1} \tag{4.12}$$

4.2 Residues

Consider the case where a function $f(z)$ is singular at $z = z_0$ but analytic for $0 < |z - z_0| < R$; we say that z_0 is an *isolated singularity*, and by Laurent's theorem we can write

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{4.13}$$

where

$$a_k = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{k+1}}\, dz \tag{4.14}$$

for any curve $\mathcal{C}$ which goes ones counter-clockwise around z_0 in the region $0 < |z - z_0| < R$. In particular,

$$a_{-1} = \frac{1}{2\pi i} \oint_{\mathcal{C}} f(z)\, dz \tag{4.15}$$

but this tells us that the integral of $f(z)$ arount an isolated singularity is just determined by the coëfficient of $(z - z_0)^{-1}$ in the Laurent series. This coëfficient is thus important enough that it has a name, "the residue of $f(z)$ at z_0". We write

$$\oint_{\mathcal{C}} f(z)\, dz = 2\pi i \operatorname{Res}(f(z), z_0) = 2\pi i a_{-1} \tag{4.16}$$

There are different ways to calculate the residue:

- If we have the Laurent series valid for $0 < |z - z_0| < R$, we can just read off a_{-1}.
- If it's a simple pole,

  $$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots \tag{4.17}$$

  so

  $$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{4.18}$$

- similarly, if z_0 is a pole of order n,

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{4.19}$$

- If $f(z) = g(z)/h(z)$ where $g(z_0) \neq 0$ and $h(z) = 0$, then

$$\text{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \tag{4.20}$$

4.2.1 The Residue Theorem

Knowing the residues of a function at all of its poles makes it very easy to integrate that function around a closed curve. This is because we can always deform the contour into a series of little loops, one around each pole, with connections between them which cancel out. The result is that, if a function has poles at $\{z_k\}$ which are inside a contour $\mathcal{C}$,

$$\oint_{\mathcal{C}} f(z)\, dz = \sum_k \oint_{\mathcal{C}_k} f(z)\, dz = 2\pi i \sum_k \text{Res}(f(z), z_k) \ . \tag{4.21}$$

This is called *Cauchy's Residue Theorem.*

Practice Problems

19.4.1, 19.4.5, 19.4.9, 19.4.13, 19.4.19, 19.5.5, 19.5.13, 19.5.17, 19.5.21, 19.5.29

Thursday 31 January 2013

5 Using Contour Integrals to Evaluate Real Integrals

Armed with Cauchy's residue theorem, we can now evaluate many different contour integrals around closed curves. As we'll now see, this in turn lets us evaluate many *real* integrals by converting them into integrals in the complex plane. These integrals may be around closed curves, or they may form a part of a closed curve.

5.1 Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

The first sort of integral is of the form

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \tag{5.1}$$

We evaluate it by making the substitution $z = e^{i\theta}$. Then by noting that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} \tag{5.2a}$$

$$\sin\theta = \frac{e^{i\theta} i e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} \tag{5.2b}$$

we can identify the integral as a parametrization of a contour integral clockwise around the unit circle, since

$$dz = ie^{i\theta}\, d\theta = iz\, d\theta \tag{5.3}$$

which means

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_{\mathcal{C}_1(0)} F\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{dz}{iz} \tag{5.4}$$

where $\mathcal{C}_1(0)$ is the unit circle $|z| = 1$.

For example,

$$\int_0^{2\pi} \frac{1}{1 + \cos\theta/2} d\theta = \oint_{\mathcal{C}_1(0)} \frac{1}{1 + (z + z^{-1})/4} \frac{dz}{iz} = \oint_{\mathcal{C}_1(0)} \frac{4\, dz}{i(z^2 + 4z + 1)} \tag{5.5}$$

Now, the integrand has poles when $z^2 + 4z + 1 = 0$, which the quadratic formula tells us are at

$$z_\pm = \frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3} \tag{5.6}$$

This means that we can factor

$$z^2 + 4z + 1 = (z - z_+)(z - z_-)\ . \tag{5.7}$$

The integrand

$$\frac{4}{i(z-z_+)(z-z_-)} \tag{5.8}$$

thus has simple poles at $z_- = -2-\sqrt{3}$ and $z_- = -2-\sqrt{3}$. These are both on the real axis. Noting that $\sqrt{3} \approx 1.73$, we see that $z_- < -1$ so it is outside the unit circle, but $-1 < z_+ < 1$ so it is inside the unit circle. This means the contour integral includes only the contribution from the residue at z_+, i.e.,

$$\oint_{\mathcal{C}_1(0)} \frac{4\,dz}{i(z^2+4z+1)} = 2\pi i \operatorname{Res}\left(\frac{4}{i(z-z_+)(z-z_-)}, z_+\right) \ . \tag{5.9}$$

Since z_+ is a simple pole we can calculate the reside using

$$\operatorname{Res}(f(z), z_0) = \lim_{z\to z_0}(z-z_0)f(z) \tag{5.10}$$

which is

$$\operatorname{Res}\left(\frac{4}{i(z-z_+)(z-z_-)}, z_+\right) = \left.\frac{4}{i(z-z_-)}\right|_{z=z_+} = \frac{4}{i(z_+-z_-)} = \frac{4}{i2\sqrt{3}} = \frac{2}{i\sqrt{3}} \tag{5.11}$$

This means

$$\int_0^{2\pi} \frac{1}{1+\cos\theta/2} d\theta = 2\pi i\left(\frac{2}{i\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}} \tag{5.12}$$

Exercise: compare this calculation to exercise 19.6.1 in Zill and Wright.

5.2 Improper Integrals

The second type of integral we're interested in is an integral of the form

$$\int_{-\infty}^{\infty} g(x)\,dx \tag{5.13}$$

where $g(x)$ has one of several useful forms (usually a rational function, or a rational function times $\cos\alpha x$ or $\sin\alpha x$). In this case, we think of the integral over x as an integral in the complex plane along the real line, which we treat as a piece of a closed contour.

An integral with plus or minus infinity in the limits of integration is called an *improper integral*, and it's defined by a limit. Strictly speaking, we're supposed to take an independent limit for each end of the integral, so

$$\int_{-\infty}^{\infty} g(x)\,dx = \lim_{R_-\to\infty}\lim_{R_+\to\infty}\int_{-R_-}^{R_+} g(x)\,dx \tag{5.14}$$

so that we can break up the integral

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{0} g(x)\,dx + \int_{0}^{\infty} g(x)\,dx \tag{5.15}$$

where each integral is defined by an independent limit. However, even if the limits at both ends of the integral don't converge, you can sometimes define a way to take both limits at once to get what is known as the *Cauchy Principal Value* for the integral.

5.2.1 Cauchy Principal Value

Consider the integral

$$\int_{-\infty}^{\infty} \frac{2x}{1+x^2} dx \tag{5.16}$$

The integrand is an odd function of x, and the range of integration looks symmetrical, so you might think the integral should be zero. But the limits associated with both ends of the integral don't converge independently, as can be checked by calculating

$$\int_0^{\infty} \frac{2x}{1+x^2} dx = \lim_{R\to\infty} \int_0^R \frac{2x}{1+x^2} dx = \lim_{R\to\infty} \left[\ln(1+x^2)\right]_0^R = \lim_{R\to\infty} \ln(1+R^2) \tag{5.17}$$

which is undefined. However, we can define something called the *Cauchy Principal Value* of the integral as

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{2x}{1+x^2} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{2x}{1+x^2} dx = \lim_{R\to\infty} \left[\ln(1+R^2) - \ln(1+R^2)\right] = 0 \tag{5.18}$$

It is this integral which we calculate with the contour integration technique, even if the standard improper integral is divergent.

Note that this is not as trivial as it sounds; for example,

$$\lim_{R\to\infty} \int_{-R}^{2R} \frac{2x}{1+x^2} dx = \lim_{R\to\infty} \left[\ln(1+4R^2) - \ln(1+R^2)\right] = \ln 4 \neq 0 \tag{5.19}$$

5.2.2 Integrals of the Form $\int_{-\infty}^{\infty} f(x)\, dx$

To evaluate the principal value of the integral

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx \tag{5.20}$$

We note that we can make a closed semicircular contour $\mathcal{C}$ by integrating along the real axis from $-R$ to R, and then counter-clockwise around the semicircle $\mathcal{C}_R$ of radius R centered at the origin. The integral of $f(z)$ around this closed loop is

$$\oint_{\mathcal{C}} f(z)\, dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx + \int_{\mathcal{C}_R} f(z)\, dz \tag{5.21}$$

As we take the limit $R \to \infty$, we enclose the entire upper half-plane in the contour integral, and we can evaluate the integral by calculating the residues at all of the poles in the upper half plane. Then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx = \sum_{\substack{z_0 \\ \text{Im}(z_0)>0}} 2\pi i \, \text{Res}(f(z), z_0) - \lim_{R\to\infty} \int_{\mathcal{C}_R} f(z)\, dz \tag{5.22}$$

So the contour integral gives us the principal value of the real integral as long as the integral around the semicircle goes to zero as $R \to \infty$. That will happen, by the ML bound, as long as the denominator of $f(z)$ has two or more powers of z than the numerator.

For example, consider

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx \; ; \tag{5.23}$$

The function $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$ has poles at $z = i$ (which is in the upper half plane) and $z = -i$ (which is in the lower half plane). The residue at the pole at $z = i$ is

$$\text{Res}\left(\frac{1}{z^2+1}, i\right) = \frac{1}{i+i} = \frac{1}{2i} \tag{5.24}$$

so if $\mathcal{C}$ is a semicircle of radius $R > 1$,

$$\oint_{\mathcal{C}} f(z)\, dz = 2\pi i \left(\frac{1}{2i}\right) = \pi = \int_{-R}^{R} f(x)\, dx + \int_{\mathcal{C}_R} f(z)\, dz \tag{5.25}$$

On $\mathcal{C}_R$,

$$|f(z)| = \frac{1}{R^2+1} \; ; \tag{5.26}$$

since the length of $\mathcal{C}_R$ is πR, that means

$$\left| \int_{\mathcal{C}_R} f(z)\, dz \right| \leq \frac{\pi R}{R^2+1} \tag{5.27}$$

which does indeed go to zero as $R \to \infty$, so the integral is

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx = \pi \tag{5.28}$$

(Of course, this integral is simple enough that we could have found the answer by trigonometric substitution, but the contour method lets us evaluate much more complicated integrals.)

The key thing to remember is that for the integral around the semicircle to vanish, you have to have *two* more powers of x in the denominator than in the numerator. For instance, the method doesn't work for

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x}{x^2+1}\, dx \tag{5.29}$$

(which we've already seen is zero), because the integral around the semicircle doesn't go to zero in that case. (In this case the integral around the closed loop is πi, all of which comes from the semicircle.)

5.2.3 Integrals of the Form $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ or $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$

If the integrand is not just a rational function, but includes $\cos\alpha x$ or $\sin\alpha x$, with $\alpha > 0$ we can use a very similar method to evaluate it. We can use the Euler formula

$$e^{i\alpha x} = \cos\alpha x + i\sin\alpha x \tag{5.30}$$

to write

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx \tag{5.31}$$

This sort of integral comes up all the time in Fourier analysis. Now, $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx$ is an integral of a complex integrand with respect to a real variable, so it has a complex result. But the integrals $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$ are integrals of real functions of a real variable, so the results must be real. We can thus read off

$$\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx = \operatorname{Re}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx \tag{5.32a}$$

$$\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx = \operatorname{Im}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx \tag{5.32b}$$

We do the integral

$$\oint_{\mathcal{C}} f(z)e^{i\alpha z}\,dz = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx + \int_{\mathcal{C}_R} f(z)e^{i\alpha z}\,dz \tag{5.33}$$

as before by looking at the residues at the poles in the upper half-plane. But now the conditions on $f(z)$ are somewhat less stringent; it's enough for the denominator to have *one* more power of z than the numerator. It's a somewhat subtle thing to show this, and it actually has a name, Jordan's lemma. We can't just use the ML bound because while

$$\left|e^{i\alpha z}\right| = \left|e^{i\alpha x}e^{-\alpha y}\right| = e^{-\alpha y} \le 1 \tag{5.34}$$

on the contour $\mathcal{C}_R$, the exponential suppression is enough to kill the integral over enough of the contour. To give a concrete example,

$$\lim_{R\to\infty}\int_{\mathcal{C}_R} \frac{z}{z^2+1}\,dz \neq 0 \tag{5.35}$$

but

$$\lim_{R\to\infty}\int_{\mathcal{C}_R} \frac{ze^{i\alpha z}}{z^2+1}\,dz = 0 \tag{5.36}$$

We can use this technique to evaluate the Fourier integrals

$$\int_{-\infty}^{\infty} \frac{x\cos kx}{x^2+a^2}\,dx \tag{5.37a}$$

$$\int_{-\infty}^{\infty} \frac{x\sin kx}{x^2+a^2}\,dx\ , \tag{5.37b}$$

where k and a are positive constants, as the real and imaginary parts of

$$\int_{-\infty}^{\infty} \frac{xe^{ikx}}{x^2 + a^2}\, dx \tag{5.38}$$

The complex function

$$\frac{ze^{ikz}}{z^2 + a^2} = \frac{ze^{ikz}}{(z - ia)(z + ia)} \tag{5.39}$$

has poles at $z = ia$ and $z = -ia$. The pole at $z = ia$ has a residue of

$$\left.\frac{ze^{ikz}}{(z + ia)}\right|_{z=ia} = \frac{iae^{-ka}}{2ia} = \frac{e^{-ka}}{2} \tag{5.40}$$

This means that

$$\oint_{\mathcal{C}} \frac{ze^{ikz}}{z^2 + a^2}\, dz = 2\pi i \left(\frac{e^{-ka}}{2}\right) = \pi i e^{-ka} \tag{5.41}$$

which means the Fourier integrals have the principal values

$$\int_{-\infty}^{\infty} \frac{x\cos kx}{x^2 + a^2}\, dx = 0 \tag{5.42a}$$

$$\int_{-\infty}^{\infty} \frac{x\sin kx}{x^2 + a^2}\, dx = \pi e^{-ka} \tag{5.42b}$$

5.2.4 Indented Contours

The idea of a Cauchy principal value of an integral also applies to integrals which diverge at some point on the real axis rather than at infinity. For example, consider

$$\int_{-\infty}^{\infty} \frac{1}{x(x^2 + 1)}\, dx \tag{5.43}$$

This integral does not converge because

$$\int_{-\infty}^{0} \frac{1}{x(x^2 + 1)}\, dx = \lim_{r\to 0+} \int_{-\infty}^{-r} \frac{1}{x(x^2 + 1)}\, dx \tag{5.44}$$

and

$$\int_{0}^{\infty} \frac{1}{x(x^2 + 1)}\, dx = \lim_{r\to 0+} \int_{r}^{\infty} \frac{1}{x(x^2 + 1)}\, dx \tag{5.45}$$

diverge logarithmically. But we can define the Cauchy principal value as

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x(x^2 + 1)}\, dx = \lim_{r\to 0+} \left(\int_{-\infty}^{-r} \frac{1}{x(x^2 + 1)}\, dx + \int_{r}^{\infty} \frac{1}{x(x^2 + 1)}\, dx\right) \tag{5.46}$$

In general, if we have a function $f(z)$ with a simple pole at $z = c \in \mathbb{R}$, we can evaluate the Cauchy principal value of its integral along the real axis with a modification of the standard contour integration technique. Let $\mathcal{C}$ be a contour made up of four parts:

- along the real line from $-R$ to $c - r$
- a *clockwise* semicircle $-\mathcal{C}_r$ of radius R centered on the origin
- along the real line from $c + r$ to R
- a counter-clockwise semicircle $\mathcal{C}_R$ of radius R centered on the origin

the integral around the closed contour $\mathcal{C}$, which we can evaluate by evaluation of the residues of $f(z)$ in the upper half-plane, is

$$\begin{aligned}\oint_{\mathcal{C}} f(z)\,dz &= \lim_{r\to 0+}\lim_{R\to\infty}\left(\int_{-R}^{c-r} f(x)\,dx + \int_{-\mathcal{C}_r} f(z)\,dz + \int_{c+r}^{R} f(x)\,dx + \int_{\mathcal{C}_R} f(z)\,dz\right) \\ &= \text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx - \lim_{r\to 0+}\int_{\mathcal{C}_r} f(z)\,dz + \lim_{R\to\infty}\int_{\mathcal{C}_R} f(z)\,dz\end{aligned} \tag{5.47}$$

We know that the integral around $\mathcal{C}_R$ will vanish in the limit $R \to \infty$ if $f(z)$ falls off fast enough (either two more powers of z in the denominator, or one more power of z in the denominator and $e^{i\alpha z}$ with α in the numerator). The contribution from the integral around $\mathcal{C}_r$ comes from the principal part of the Laurent series for $f(z)$ centered at c. Since we're assuming $f(z)$ has a simple pole, we can write the Laurent series as

$$f(z) = \frac{a_{-1}}{z-c} + g(z) \tag{5.48}$$

where $a_{-1} = \text{Res}(f(z), c)$ and $g(z)$ is analytic in a neighborhood around $z = c$. It's easy to use the ML bound to show

$$\lim_{r\to 0+}\int_{\mathcal{C}_r} g(z)\,dz = 0 \tag{5.49}$$

so

$$\lim_{r\to 0+}\int_{\mathcal{C}_r} f(z)\,dz = \lim_{r\to 0+}\int_{\mathcal{C}_r} \frac{a_{-1}}{z-c}\,dz \tag{5.50}$$

If we parameterize $\int_{\mathcal{C}_r}$ as

$$z = c + re^{i\theta} \tag{5.51}$$

$$\theta_I = 0 \tag{5.52}$$

$$\theta_F = \pi \tag{5.53}$$

so that

$$dz = ire^{i\theta}\,d\theta \tag{5.54}$$

we have

$$\int_{\mathcal{C}_r} \frac{a_{-1}}{z-c}\,dz = \int_0^{\pi} \frac{a_{-1}}{re^{i\theta}} ire^{i\theta}\,d\theta = ia_{-1}\int_0^{\pi} d\theta = i\pi a_{-1} = i\pi\,\text{Res}(f(z), c) \tag{5.55}$$

So that means that

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \oint_{\mathcal{C}} f(z)\,dz + \lim_{r\to 0+} \int_{\mathcal{C}_r} f(z)\,dz = 2\pi i \sum_{\substack{z_0 \\ \operatorname{Im}(z_0)>0}} \operatorname{Res}(f(z), z_0) + \pi i \operatorname{Res}(f(z), c) \tag{5.56}$$

This makes some intuitive sense; you pick up half of the residue of a pole on the real axis. Note that this only works for simple poles, however.

Example: Find

$$\text{P.V} \int_{-\infty}^{\infty} \frac{1}{z^4-1}\,dz \tag{5.57}$$

The integrand

$$\frac{1}{z^4-1} = \frac{1}{(z+i)(z-i)(z+1)(z-1)} \tag{5.58}$$

has poles at the four fourth roots of unity. The residues are

$$\operatorname{Res}\left(\frac{1}{z^4-1}, 1\right) = \frac{1}{(1+i)(1-i)(1+1)} = \frac{1}{(2)(2)} = \frac{1}{4} \tag{5.59a}$$

$$\operatorname{Res}\left(\frac{1}{z^4-1}, i\right) = \frac{1}{(i+i)(i+1)(i-1)} = \frac{1}{(2i)(-2)} = -\frac{1}{4i} \tag{5.59b}$$

$$\operatorname{Res}\left(\frac{1}{z^4-1}, -1\right) = \frac{1}{(-1+i)(-1-i)(-1-1)} = \frac{1}{(2)(-2)} = -\frac{1}{4} \tag{5.59c}$$

$$\operatorname{Res}\left(\frac{1}{z^4-1}, -i\right) = \frac{1}{(-i-i)(-i+1)(-i-1)} = \frac{1}{(-2i)(-2)} = \frac{1}{4i} \tag{5.59d}$$

The contour integral picks up the residue at $z = i$ and half of the residues at $z = 1$ and $z = -1$, so

$$\begin{aligned} \text{P.V} \int_{-\infty}^{\infty} \frac{1}{z^4-1}\,dz &= \pi i \left[2 \operatorname{Res}\left(\frac{1}{z^4-1}, i\right) + \operatorname{Res}\left(\frac{1}{z^4-1}, -1\right) + \operatorname{Res}\left(\frac{1}{z^4-1}, 1\right)\right] \\ &= -\frac{\pi}{2} - \frac{i\pi}{4} + \frac{i\pi}{4} = -\frac{\pi}{2} \end{aligned} \tag{5.60}$$

Practice Problems

19.6.3, 19.6.5, 19.6.9, 19.6.11, 19.6.13, 19.6.21, 19.6.23, 19.6.29, 19.6.31