STAT 405-01: Mathematical Statistics I

Problem Set 11

Assigned 2013 November 26
Due 2013 December 3

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Hogg 4.7.3

2 Hogg 4.7.7

3 Hogg 4.7.9

4 Hogg 5.1.2

5 Extra Credit

Carry out the following steps to demonstrate that if $\mathbf{X} = (X_1, \cdots, X_k)$ is a multinomial random vector with n trials, probabilities $p_1, \ldots, p_k$ where $\sum_{i=1}^{k} p_i = 1$, and all of the $\{np_i\}$ are large enough that we can approximate $\mathbf{X}$ as a multivariate normal random vector with the same mean and variance-covariance matrix, then the statistic $Q = \sum_{i=1}^{k} \frac{(X_i - np_i)^2}{np_i}$ approximately obeys a $\chi^2(k-1)$ distribution.

(a) The joint mgf for all k multinomial random variables is $M_{\mathbf{X}}(t_1, \ldots, t_k) = (p_1\, e^{t_1} + \cdots + p_k\, e^{t_k})^n$. (This looks slightly different than the one given in the text, which was the distribution for the first $k-1$ of the variables, since the constraint $\sum_{i=1}^{k} X_i = n$ means that X_k is determined by the other random variables.) Use this to find the mean $\boldsymbol{\mu} = E(\mathbf{X})$ and variance-covariance matrix $\boldsymbol{\Sigma} = \mathrm{Cov}(\mathbf{X})$. (The latter can be found by calculating a typical diagonal element like $\mathrm{Var}(X_1)$ and a typical off-diagonal element $\mathrm{Cov}(X_1, X_2)$ and generalizing.)

(b) Assume that we can treat $\mathbf{X}$ approximately as a $N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ multivariate normal random vector. Define a random vector $\mathbf{Y} = (\frac{X_1 - np_1}{\sqrt{np_1}}, \cdots, , \frac{X_k - np_k}{\sqrt{np_k}})$, and by writing $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, use theorem 3.5.1 to show that $\mathbf{Y}$ can be treated approximately as a $N_k(\mathbf{0}, \mathbf{1} - \mathbf{w}\mathbf{w}^{\mathrm{T}})$ multivariate normal random vector, where $\mathbf{w} = (\sqrt{p_1}, \cdots, \sqrt{p_k})$

(c) Show that $\mathbf{w}^{\mathrm{T}}\mathbf{w} = 1$ and use this to show that $\mathbf{1} - \mathbf{w}\mathbf{w}^{\mathrm{T}}$ is a projector onto the $k-1$ dimensional subspace perpendicular to $\mathbf{w}$. Use this to show that, analogous to the proof of point 3 of Student's theorem given in the lecture notes, $Q = \mathbf{Y}^{\mathrm{T}}\mathbf{Y}$ can be treated approximately as a $\chi^2(k-1)$ random variable.