ASTP 611-01: Statistical Methods for Astrophysics

Problem Set 1

Assigned 2014 January 29
Due 2014 February 4

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

0 Conventions

The convention we're using for the continuous Fourier transform is

$$\mathcal{F}\{h\} = \widetilde{h}(f) = \int_{-\infty}^{\infty} h(t)\, e^{-i2\pi f t}\, dt \tag{0.1a}$$

$$\mathcal{F}^{-1}\left\{\widetilde{h}\right\} = h(t) = \int_{-\infty}^{\infty} \widetilde{h}(f)\, e^{i2\pi f t}\, df \tag{0.1b}$$

and for the discrete Fourier transform

$$\widehat{h}_k = \sum_{j=0}^{N-1} h_j\, e^{-i2\pi jk/N} \tag{0.2a}$$

$$h_j = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{h}_k\, e^{i2\pi jk/N} \tag{0.2b}$$

You may find it useful to apply the identities

$$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2} \qquad \text{and} \qquad \sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i} \tag{0.3}$$

1 Parseval's Theorem

a) Use

$$\int_{-\infty}^{\infty} e^{-i2\pi(f-f')t}\, dt = \delta(f-f') = \delta(f'-f) \tag{1.1}$$

to show that

$$\int_{-\infty}^{\infty} g^*(t)\, h(t)\, dt = \int_{-\infty}^{\infty} \widetilde{g}^*(f)\, \widetilde{h}(f)\, df \tag{1.2}$$

b) Use

$$\sum_{k=0}^{N-1} e^{i2\pi(j-\ell)k/N} = N\, \delta_{j,\ell \bmod N} \tag{1.3}$$

to show that

$$\sum_{j=0}^{N-1} g_j^*\, h_j = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{g}_k^*\, \widehat{h}_k \tag{1.4}$$

2 Cosine-Gaussian Pulse

2.1 Continuous Fourier Transform

A cosine-Gaussian signal has the form

$$h(t) = h_0 \, e^{-\alpha t^2} \cos 2\pi f_1 t \tag{2.1}$$

Using the identity

$$\int_{-\infty}^{\infty} e^{-at^2+ibt}\, dt = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{b^2}{4a}\right) \qquad a > 0 \tag{2.2}$$

show that the continuous Fourier transform is

$$\widetilde{h}(f) = \frac{h_0}{2}\sqrt{\frac{\pi}{\alpha}} \left(e^{-\pi^2(f+f_1)^2/\alpha} + e^{-\pi^2(f-f_1)^2/\alpha}\right) \tag{2.3}$$

2.2 Discrete Fourier Transform

For this part of the problem, you will make use of the ipython notebook at
`http://ccrg.rit.edu/~whelan/courses/2014_1sp_ASTP_611/data/ps01.ipynb`

a) Execute each cell of the notebook. and print the results. (You need to use File>Save followed by File>Print View, and then use your web browser to print the resulting page.) Make sure you understand what is being done at each step.

b) Change the parameters `N`, `f1` and `alpha`. Can you produce situations where the discrete approximation to the continuous Fourier transform fails, by varying each one by one? What assumptions are you violating each time? For this part of the problem, you should also submit printouts of the final version of your work. The best method is if you append all three of the new computations to the original notebook, including whichever plots, zoomed plots, etc are necessary to show the breakdown of the appeoximation. If you change the cells in the original notebook instead, be sure to remove or change any of the documentation which doesn't apply.

3 Sampling and Aliasing

a) Consider the 3 Hz cosine wave $h(t) = \cos(2\pi[3\,\mathrm{Hz}]t)$, sampled at $\delta t = 0.25\,\mathrm{s}$. Sketch the continuous function $h(t)$ from $t = 0$ to $t = 2.0\,\mathrm{s}$, and put dots at the samples $h_j = h(j\,\delta t)$.

b) $\{h_j\}$ is also the discretization of a lower-frequency trigonometric function, i.e., $h_j = \mathfrak{h}(j\delta t)$. What is $\mathfrak{h}(t)$?

c) By writing a formula for h_j as a sum of complex exponentials, work out a set of Fourier components $\{\widehat{h}_k | k = 0, \ldots, 7\}$ such that h_j is given by (0.2b). (Note that this is easier and more enlightening than calculating the forward Fourier transform (0.2a) analytically or numerically for each value of k.)

d) Use the symmetries of the discrete Fourier transform to work out a set of Fourier components $\{\widehat{h}_k | k = -4, \ldots, 3\}$?