ASTP 611-01: Statistical Methods for Astrophysics

Problem Set 2

Assigned 2014 February 4
Due 2014 February 11

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Colored Noise

For this problem, you will make use of the ipython notebook at
`http://ccrg.rit.edu/~whelan/courses/2014_1sp_ASTP_611/data/ps02.ipynb`

a) Execute each cell of the notebook. and print the results. (You need to use File>Save followed by File>Print View, and then use your web browser to print the resulting page.) Make sure you understand what is being done at each step.
b) Given the parameters chosen and the calculations from class, what should be the expectation value $\left\langle |\widehat{x}_k|^2 \right\rangle$ be? Is that consistent with the observed mean $\frac{1}{N}\sum_{k=0}^{N} |\widehat{x}_k|^2$ of the actual random values? Answer the same question for the standard deviation.
c) Show that the PSD associated with the "red noise" model is $\delta t\, e^{-f_k^2/\sigma_f^2}$ (which is the "Theory" curve in the plot labelled "PSD estimates for red noise").
d) Define "blue noise" by $\widehat{z}_k = \widehat{x}_k \exp(-\frac{(|f_k|-f_c)^2}{2\sigma_f^2})$ where $\sigma_f = 16\,\text{Hz}$ and $f_c = 128\,\text{Hz}$, and re-do the plots of $|\widehat{z}_k|$ vs f_k, the inverse Fourier transform z_j vs $t_j - t_0$ and the power spectrum estimates for this "Blue noise". (I.e., add three additional plots to the notebook which do the same thing for blue noise that you did for red noise.)

2 Power Spectral Density

Consider a single random variable ψ which is equally likely to fall anywhere between 0 and 2π, so that expectation values of random variables whose only randomness comes from ψ can be calculated as

$$\langle F(\psi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} F(\psi)\, d\psi \ . \tag{2.1}$$

Let $x(t) = A\sin(2\pi f_0 t + \psi)$ for some fixed A and f_0.

a) Find $\langle x(t)\rangle$ and $\langle x(t)x(t')\rangle$ and show that $x(t)$ is wide-sense stationary.
b) Find the power spectral density $P_x(f)$.

3 Windowing in PSD Estimates

(Note: In this problem you should repeat analogous calculations from the class notes as necessary, rather than just quoting the results.) The problem of spectral leakage can be reduced by multiplying the time series $\{h_j = h(t_0 + j\,\delta t)\}$ by a window function $\{w_j\}$ before performing the discrete Fourier transform:

$$\widehat{h}_k^w = \sum_{j=0}^{N-1} w_j\, h_j\, e^{-i2\pi jk/N} \;. \tag{3.1}$$

a) Use the inverse Fourier transform

$$h(t) = \int_{-\infty}^{\infty} \widetilde{h}(f)\, e^{i2\pi f(t-t_0)}\, df \tag{3.2}$$

to find an expression for discrete Fourier transform of the windowed data of the form

$$\widehat{h}_k^w = \int_{-\infty}^{\infty} \Lambda([f_k - f]\delta t)\, \widetilde{h}(f)\, df \tag{3.3}$$

where $\Lambda(x)$ is some function constructed using the N-point window $\{w_j\}$, which you will determine.

b) Use (3.3) to find the expectation value $\left\langle \left|\widehat{h}_k^w\right|^2 \right\rangle$ in terms of the PSD $S_h(f)$ and the function $\Lambda(x)$.

c) Use the fact that, for integer j and ℓ,

$$\int_{-1/2}^{1/2} e^{-i2\pi(j-\ell)x}\, dx = \delta_{j\ell} \tag{3.4}$$

to evaluate

$$\int_{-1/2}^{1/2} |\Lambda(x)|^2\, dx \tag{3.5}$$

in terms of the mean square window value

$$\overline{w^2} = \frac{1}{N} \sum_{j=0}^{N-1} (w_j)^2 \tag{3.6}$$

d) Assuming that $|\Lambda(x)|^2$ is sharply enough peaked to be treated as a sum of approximate Dirac delta functions

$$|\Lambda(x)|^2 \approx \mathcal{A} \sum_{s=-\infty}^{\infty} \delta(x+s) \tag{3.7}$$

find the proportionality constant $\mathcal{A}$.

e) Using the approximation (3.7) (with the explicit value you found for $\mathcal{A}$), write an approximate expression for the expectation value $\langle |\widehat{h}_k^w|^2 \rangle$ in terms of $S_h(f_k)$, δt, N, and $\overline{w^2}$.

f) Use the result of part f) to write a windowed periodogram P_k^w constructed from $|\widehat{h}_k^w|^2$ such that $\langle P_k^w \rangle \approx S_h(f_k)$.