ASTP 611-01: Statistical Methods for Astrophysics

Problem Set 3

Assigned 2014 February 11
Due 2014 February 18

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Exercises in Logic and Probability

Do all of the problems at the end of Chapter 2 of Gregory.

2 Cumulant Generating Function

Given the moment generating function $M(t) = \left\langle e^{tX} \right\rangle$ from which you can calculate $\left\langle X^k \right\rangle = M^{(k)}(0)$, and in particular $M(0) = 1$, $M'(0) = \langle X \rangle = \mu_X$, and $M''(0) = \langle X^2 \rangle$, it is often convenient to define the cumulant generating function $\psi(t) = \ln M(t)$.

a) Show that $\psi(0) = 0$ and $\psi'(0) = \langle X \rangle$.
b) Show that $\psi''(0) = \mathrm{Var}(X)$.
c) Consider two random variables with a joint mgf $M(t_1, t_2) = \left\langle e^{t_1 X_1 + t_2 X_2} \right\rangle$ which can be used to find

$$\left\langle {X_1}^{k_1} {X_2}^{k_2} \right\rangle = \left. \frac{\partial^{k_1+k_2}}{\partial^{k_1} t_1 \partial^{k_2} t_2} M(t_1, t_2) \right|_{(t_1,t_2)=(0,0)} \tag{2.1}$$

Define $\psi(t_1, t_2) = \ln M(t_1, t_2)$ and show that the covariance of X_1 and X_2 can be evaluated as

$$\mathrm{Cov}(X_1, X_2) = \left. \frac{\partial^2 \psi}{\partial t_1 \, \partial t_2} \right|_{(t_1,t_2)=(0,0)} \tag{2.2}$$

[additional problem on back]

3 Change of Variables

Define traditional spherical coördinates (θ, ϕ), with θ being the angle down from the zenith (so that $\theta = 0$ is the zenith and $\theta = \pi/2$ is the horizon) and ϕ being an azimuthal angle which runs from 0 to 2π. Consider an event which occurs at a random sky location above the horizon. The joint pdf for the random variables Θ and Φ is

$$f_{\Theta\Phi}(\theta, \phi) = \frac{\sin\theta}{2\pi} \qquad 0 \le \theta \le \pi/2;\; 0 \le \phi < 2\pi \tag{3.1}$$

a) Integrate over θ and ϕ to confirm that $f_{\Theta\Phi}(\theta, \phi)$ is a normalized density in those variables.

b) Explain why (3.1) represents an isotropic probability distribution.

c) Define new random variables

$$N_x = \sin\Theta \cos\Phi \tag{3.2a}$$
$$N_y = \sin\Theta \sin\Phi \tag{3.2b}$$

be the projections onto two horizontal directions of the unit vector pointing to the event. Perform a change of variables to obtain the joint pdf $f_{N_x N_y}(n_x, n_y)$ for those two variables. (Your answer should not contain θ or ϕ, although they may be convenient for intermediate steps.)

d) What is the region of the (n_x, n_y) plane which corresponds to $0 \le \theta \le \pi/2$, $0 \le \phi < 2\pi$?

e) Marginalize over n_y to find $f_{N_x}(n_x)$.