ASTP 611-01: Statistical Methods for Astrophysics

Problem Set 11

Assigned 2014 May 6
Due 2014 May 13

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Prior Probabilities for the Binomial Distribution

Consider the binomial distribution with n trials and a probability parameter $\theta \in [0, 1]$, which has a pmf of

$$p(k|\theta, I) = \frac{n!}{(n-k)!k!}\theta^k(1-\theta)^{n-k} \qquad k = 0, 1, \ldots, n \tag{1.1}$$

a) Show that the beta distribution

$$f_{\mathrm{B}}(\theta|\alpha, \beta) = \frac{1}{\mathrm{B}(\alpha, \beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \qquad 0 < \theta < 1 \tag{1.2}$$

with parameters α and β is the conjugate prior distribution for the binomial distribution, i.e., if the prior pdf on θ is $f(\theta|I) = f_{\mathrm{B}}(\theta|\alpha, \beta)$ for some α and β, then the posterior is $f(\theta|k, I) = f_{\mathrm{B}}(\theta|\alpha', \beta')$, a beta distribution with some other parameters α' and β', which you will determine. Note that (1.2) includes the beta function

$$\mathrm{B}(\alpha, \beta) = \int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1}\, d\theta = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.3}$$

b) The maximum entropy prior for θ is uniform between 0 and 1. Show that this is a member of the conjugate prior family. What are the α and β in this case?

c) Defining the likelihood $\ell(\theta; k) = \ln p(k|\theta, I)$, calculate the Fisher information $\mathcal{I}(\theta) = -\langle \ell''(\theta; K)\rangle$ and use this to construct the Jeffreys prior $f(\theta) \propto \sqrt{\mathcal{I}(\theta)}$. Show that this is also a member of the conjugate prior family, and specify the associated α and β parameters.

d) Suppose that the prior $f(\theta|I)$ is a member of the conjugate prior family. Marginalize over θ to obtain the pmf $p(k|I)$.

e) What is $p(k|I)$ if the prior $f(\theta|I)$ is uniform?

2 Maximum Entropy with Multiple Choices

Suppose we are rolling a six-sided die and counting the number of 1s. If we wish to express ignorance about all properties of the die except that it has m sides and doesn't change its properties from roll to roll, we should use a multinomial distribution with probabilities $\{\theta_i | i = 1, \ldots, 6\}$ to describe the experiment, and use the maximum entropy prior, which is uniform over the space of allowed $\{\theta_i\}$ values:

$$f(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = \begin{cases} C_6 & \begin{array}{l} 0 < \theta_1 < 1,\, 0 < \theta_2 < 1 - \theta_1,\, 0 < \theta_3 < 1 - \theta_1 - \theta_2, \\ 0 < \theta_4 < 1 - \theta_1 - \theta_2 - \theta_3,\, 0 < \theta_5 < 1 - \theta_1 - \theta_2 - \theta_3 - \theta_4 \end{array} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

(We don't need to make the pdf a function of θ_6 because θ_6 must equal $1 - \theta_1 - \theta_2 - \theta_3 - \theta_4 - \theta_5$.)

a) Marginalize over θ_5, θ_4, θ_3, and θ_2 to obtain a prior $f(\theta_1)$.
b) By requiring $f(\theta_1)$ to be normalized, find the constant C_6 appearing in (2.1).
c) Generalize your result to the case where the die has m sides, where m is any positive integer.