STAT 406-01: Mathematical Statistics II

Problem Set 12

Assigned 2016 May 3
Due 2016 May 10

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Hogg 8.3.5

2 Hogg 8.3.6

3 Hogg 8.3.7

4 Hogg 8.3.10

5 Bayesian Composite Hypothesis Testing

Consider a single random variable with pdf $f(x;\theta) = \theta e^{-\theta x}$, $0 < x < \infty$. We wish to test the point hypothesis $\mathcal{H}_0 : \theta = 1$ against the composite hypothesis $\mathcal{H}_1$ with prior distribution $f_\Theta(\theta|\mathcal{H}_1) = (\epsilon_2^{\epsilon_1}/\Gamma(\epsilon_1))\theta^{\epsilon_1 - 1}e^{-\epsilon_2\theta}$, $0 < \theta < \infty$, $0 < \epsilon_1 < \infty$, $0 < \epsilon_2 < \infty$.

(a) Calculate the power function $\gamma(\theta)$ for a test which rejects $\mathcal{H}_0$ if $X \leq c_1$ or $X \geq c_2$ for some $0 \leq c_1 < c_2 < \infty$.

(b) Calculate the significance α of such a test as a function of c_1 and c_2.

(c) Calculate the power $\gamma(\mathcal{H}_1) = \int_0^\infty \gamma(\theta)\, f_\Theta(\theta|\mathcal{H}_\infty)\, d\theta$ of this test under the hypothesis $\mathcal{H}_1$ including the prior distribution on θ.

(d) Find the maximum likelihood estimate $\widehat{\theta}(x)$ on the parameter space $0 < \theta < \infty$.

(e) Construct the likelihood ratio statistic for a test of $\mathcal{H}_0$ versus an alternative in which $0 < \theta < \infty$ and find an equation relating c_1 and c_2 for a likelihood ratio test.

(f) Construct the Bayes factor for $\mathcal{H}_0$ vs $\mathcal{H}_1$ and find an equation relating c_1 and c_2 for the optimal Bayesian test. Comment on the form of the test in the limit that ϵ_1 and ϵ_2 both go to zero.

(g) (Extra credit) For the case $\epsilon_1 = 1 = \epsilon_2$, numerically evaluate c_1 and c_2 for both the likelihood ratio test and the Bayes-factor test, if the significance of the test is $\alpha = 0.05$. Plot the power function $\gamma(\theta)$ and note whether each of the tests is unbiased. Determine the marginalized power $\gamma(\mathcal{H}_1)$ for each test, and note which one is higher.