STAT 489-01: Bayesian Methods of Data Analysis

Problem Set 6

Assigned 2017 March 9
Due 2017 March 23

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Bayes Factor for Counting Experiment

Return to the data considered in class Tuesday, March 7, available from
`http://ccrg.rit.edu/~whelan/courses/2017_1sp_STAT_489/data/notes_models_poisproc.dat`
Consider a model M'' which contains four different Poisson rates, $\{\theta_1, \theta_2, \theta_3, \theta_4\}$, which apply from 00:00 to 06:00, 06:00 to 12:00, 12:00 to 18:00, and 18:00 to 24:00, respectively. Calculate the evidence for this model and calculate the Bayes factors $\mathcal{B}_{M'',M}$ and $\mathcal{B}_{M'',M'}$ (or their logarithms) with the two models considered in class.

2 Marginal Posteriors for Regression

Consider the likelihood

$$p(\mathbf{y}|\mathbf{x}, \alpha, \beta, \sigma^{-2}, M_1, I) = \left(\frac{\sigma^{-2}}{2\pi}\right)^{n/2} \exp\left(-\frac{\sigma^{-2}}{2}\left[S_{yy} - \frac{S_{xy}^2}{S_{xx}} + S_{xx}\left(\beta - \frac{S_{xy}}{S_{xx}}\right)^2 + n(\overline{y} - \alpha)^2\right]\right) \tag{2.1}$$

derived in class for the linear regression model. Assume a non-informative prior

$$p(\alpha, \beta, \sigma^{-2}|M_1, I) \propto 1/\sigma^{-2}, \qquad -\infty < \alpha, \beta < \infty;\ 0 < \sigma^{-2} < \infty\ . \tag{2.2}$$

(a) Marginalize over σ^{-2} to obtain the posterior $p(\alpha, \beta|\mathbf{y}, \mathbf{x}, M_1, I)$.

(b) Marginalize over α and β to obtain the posterior $p(\sigma^{-2}|\mathbf{y}, \mathbf{x}, M_1, I)$. From the properties of the distribution, propose an estimator for σ based on either the posterior espectation or MAP estimate of σ^{-2}.

(c) Plot the marginal posteriors $p(\alpha|\mathbf{y}, \mathbf{x}, M_1, I)$ $p(\beta|\mathbf{y}, \mathbf{x}, M_1, I)$, and $p(\sigma^{-2}|\mathbf{y}, \mathbf{x}, M_1, I)$ where $\mathbf{x}$ is the height in cm and $\mathbf{y}$ is the weight in kg for the adults in the `Howell1` dataset considered in class.

3 Bayes Factor for Regression Experiment

(a) Modify the improper prior (2.2) to be a normalized distribution with support only for $\alpha_{\text{min}} < \alpha < \alpha_{\text{max}}$, $\beta_{\text{min}} < \beta < \beta_{\text{max}}$, and $\sigma_{\text{min}} < \sigma < \sigma_{\text{max}}$, and calculate the evidence $p(\mathbf{y}|\mathbf{x}, M_1, I)$ for model M_1 as a function of the data $\mathbf{x}$, $\mathbf{y}$.

(b) Calculate the log Bayes factor $\ln \mathcal{B}_{10}$ between the two models M_1 and M_0 for the adult heights and weights in the `Howell1` dataset. Use a prior for which $\beta_{\text{min}} = -\beta_{\text{max}}$. A reasonably conservative choice for β_{max} would be $\mu_{\text{max}} - \mu_{\text{min}}$ (from class) divided by the range of the $\{x_i\}$.

(c) Now remove the restriction `Howell1$age >= 18` and scatter plot the heights and weights; does it look like a single linear model will describe the data adequately?

(d) Calculate the Bayes factor between $\mathcal{M}_1$ applied to the entire dataset and a model $\mathcal{M}_{1+1}$ in which one linear model with parameters α_1, β_1, σ_1 is fit to the adult height and weight data, and a second model with parameters α_2, β_2, σ_2 is fit to the data for which `Howell1$age < 18`.