# ASTP 611-01: Statistical Methods for Astrophysics

Problem Set 4

Assigned 2017 September 19
Due 2017 September 26

**Show your work on all problems!** Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

## 1 Cumulant Generating Function, cont'd

Consider two random variables with a joint mgf $M(t_1, t_2) = E\left[e^{t_1 X_1 + t_2 X_2}\right]$ which can be used to find

$$E\left[{X_1}^{k_1} {X_2}^{k_2}\right] = \frac{\partial^{k_1+k_2}}{\partial^{k_1} t_1 \partial^{k_2} t_2} M(t_1, t_2)\bigg|_{(t_1,t_2)=(0,0)} \tag{1.1}$$

Define $\psi(t_1, t_2) = \ln M(t_1, t_2)$ and show that the covariance of $X_1$ and $X_2$ can be evaluated as

$$\operatorname{Cov}(X_1, X_2) = \frac{\partial^2 \psi}{\partial t_1\, \partial t_2}\bigg|_{(t_1,t_2)=(0,0)} \tag{1.2}$$

## 2 Change of Variables

Define traditional spherical coördinates $(\theta, \phi)$, with $\theta$ being the angle down from the zenith (so that $\theta = 0$ is the zenith and $\theta = \pi/2$ is the horizon) and $\phi$ being an azimuthal angle which runs from 0 to $2\pi$. Consider an event which occurs at a random sky location above the horizon. The joint pdf for the random variables $\Theta$ and $\Phi$ is

$$f_{\Theta\Phi}(\theta, \phi) = \frac{\sin\theta}{2\pi} \qquad 0 \le \theta \le \pi/2;\ 0 \le \phi < 2\pi \tag{2.1}$$

a) Integrate over $\theta$ and $\phi$ to confirm that $f_{\Theta\Phi}(\theta, \phi)$ is a normalized density in those variables.

b) Explain why (2.1) represents an isotropic probability distribution.

c) Define new random variables

$$N_x = \sin\Theta \cos\Phi \tag{2.2a}$$
$$N_y = \sin\Theta \sin\Phi \tag{2.2b}$$

be the projections onto two horizontal directions of the unit vector pointing to the event. Perform a change of variables to obtain the joint pdf $f_{N_x N_y}(n_x, n_y)$ for those two variables. (Your answer should not contain $\theta$ or $\phi$, although they may be convenient for intermediate steps.)

d) What is the region of the $(n_x, n_y)$ plane which corresponds to $0 \le \theta \le \pi/2$, $0 \le \phi < 2\pi$?

e) Marginalize over $n_y$ to find $f_{N_x}(n_x)$.

# 3 Poisson Distribution

Consider a Poisson process with rate $r$. Let $k_1$ be the number of events occurring between $t = 0$ and $t = T_1$ and $k_2$ be the number of events occurring between $t = 0$ and $t = T_2$, where $0 < T_1 < T_2$.

a) Use the Poisson distribution to find the following probability mass functions (where $I$ is the background information involved in setting up the problem but does *not* include the specification of the values of $r$, $k_1$ or $k_2$):

   i) The pmf $p(k_1|r, I)$ for the number of events between $t = 0$ and $t = T_1$. For which values of $k_1$ is it non-zero?

   ii) The pmf $p(k_2|r, I)$ for the number of events between $t = 0$ and $t = T_2$. For which values of $k_2$ is it non-zero?

   iii) The pmf $p([k_2 - k_1]|r, I)$ for the number of events between $t = T_1$ and $t = T_2$. For which values of $k_2 - k_1$ is it non-zero?

   iv) The conditional pmf $p(k_2|k_1, r, I)$ for the number of events between $t = 0$ and $t = T_2$, given that $k_1$ events occurred between $t = 0$ and $t = T_1$, where $k_1$ is a non-negative integer. For which values of $k_2$ is it non-zero?

b) Use Bayes's theorem to find the conditional pmf $p(k_1|k_2, r, I)$ for the number of events between $t = 0$ and $t = T_1$, given that $k_2$ events occurred between $t = 0$ and $t = T_2$, where $k_2$ is a non-negative integer. Simplify your result as much as possible. For which values of $k_1$ is it non-zero?

c) Your result for part b) should have the form of a binomial distribution. Describe an alternate derivation of the pmf $p(k_1|k_2, r, I)$ using the properties of a Poisson process which leads directly to the binomial form.

# 4 Trinomial Distribution

Consider random variables $X_1$, $X_2$, and $X_3$, which obey a trinomial distribuition

$$p(x_1, x_2, x_3) = \begin{cases} \frac{n!}{x_1!x_2!x_3!}\theta_1^{x_1}\theta_2^{x_2}\theta_3^{x_3}, & x_1 = 0, 1, \ldots, n;\ x_2 = 0, 1, \ldots, n - x_1;\ x_3 = n - x_1 - x_2; \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

where $\{\theta_l\}$ are non-negative parameters with $\theta_1 + \theta_2 + \theta_3 = 1$.

a) Look up the multinonial theorem and use it to show that the moment-generating function is $M(t_1, t_2, t_3) := E\left[e^{t_1 X_1 + t_2 X_2 + t_3 X_3}\right] = (\theta_1\, e^{t_1} + \theta_2\, e^{t_2} + \theta_3\, e^{t_3})^n$

b) Use the mgf to calculate $E[X_1]$, $\text{Var}(X_1)$, and $\text{Cov}(X_1, X_2)$.