ASTP 611-01: Statistical Methods for Astrophysics

Problem Set 10

Assigned 2017 November 21
Due 2017 November 30

Show your work on all problems! Be sure to give credit to any collaborators, or outside sources used in solving the problems. Note that if using an outside source to do a calculation, you should use it as a reference for the method, and actually carry out the calculation yourself; it's not sufficient to quote the results of a calculation contained in an outside source.

1 Counting Experiment with Jeffreys Prior

a) Consider the problem of a counting experiment where the prior on the rate r is not uniform, but rather

$$f(r|I) = \begin{cases} \frac{1}{\ln(r_{\max}/r_{\min})}\,\frac{1}{r} & r_{\min} < r < r_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

   i) Work out the posterior $f(r|k,I)$ after a Poisson experiment in which $k > 0$ events are seen in an observing time T, assuming that $0 < r_{\min} \ll \frac{1}{T}$ and $k \ll r_{\max}T$.
   ii) Work out the posterior for $k = 0$ under the same circumstances.
   iii) Plot the posterior pdf for the case where $k = 5$. (You don't need to specify the value of T if you plot $\frac{f(r|k,I)}{T}$ versus rT.)

b) Suppose now that there is a known background event rate b, and the signal rate s has a prior

$$f(s|I) = \begin{cases} \frac{1}{\ln(s_{\max}/s_{\min})}\,\frac{1}{s} & s_{\min} < s < s_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

   i) Work out the posterior $f(r|k,I)$ after a Poisson experiment in which $k > 0$ events are seen in an observing time T, assuming that $0 < s_{\min} \ll \frac{1}{T}$ and $k \ll s_{\max}T$
   ii) Work out the posterior for $k = 0$ under the same circumstances.
   iii) Plot the posterior pdf for the case where $b = \frac{4.1}{T}$ and $k = 9$. (You don't need to specify the value of T if you plot $\frac{f(s|k,I)}{\mathcal{N}T}$ versus sT.)

2 Counting experiments with unknown background rate

Use the ipython notebook `http://ccrg.rit.edu/~whelan/courses/2017_3fa_ASTP_611/data/ps10.ipynb` to investigate the rate posteriors arising from counting experiments. Wherever there is an **EXERCISE**, add code at that point in the notebook to perform the requested calculations.

3 Flip-Flopping

Recall Problem 3 on Problem Set 8, on which you found that if a measurement x of a parameter θ was considered to be a realization of a random variable X with pdf

$$f(x|\theta) = \frac{e^{-(x-\theta)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \tag{3.1}$$

the 90% frequentist upper limit on θ was

$$u_1(x) = x + z_{0.10} \tag{3.2}$$

where z_α is defined by

$$\alpha = \int_{z_\alpha}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = \int_{-\infty}^{-z_\alpha} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz \tag{3.3}$$

And $z_{0.10} \approx 1.28$. (Note that in that problem I used α for the confidence level, but in this context it is the "tail probability" and the CL for the upper limit is $1-\alpha$.) This is defined so that, for any θ,

$$0.90 = P(\theta \le u_1(X)|\theta) = \int_{u_1^{-1}(\theta)}^{\infty} f(x|\theta)\,dx \tag{3.4}$$

where u_1^{-1} is the inverse function of u_1, defined by $u_1^{-1}(u(x)) = x$.

a) Construct the 90% frequentist two-sided confidence interval $[\ell_2(x), u_2(x)]$ on θ, defined such that

$$0.90 = P(\ell_2(X) \le \theta \le u_2(X)|\theta) = \int_{u_2^{-1}(\theta)}^{\ell_2^{-1}(\theta)} f(x|\theta)\,dx \tag{3.5}$$

and specifically

$$0.05 = P(\theta < \ell_2(X)|\theta) = \int_{\ell_2^{-1}(\theta)}^{\infty} f(x|\theta)\,dx = P(u_2(X) < \theta|\theta) = \int_{-\infty}^{u_2^{-1}(\theta)} f(x|\theta)\,dx \tag{3.6}$$

b) Consider the case of the "flip-flopping physicist"[1] who decides, since $\theta \ge 0$ on physical grounds, to do the following:

i) if $x > z_{0.05}\sigma$, construct a two-sided interval confidence interval $[\ell_2(x), u_2(x)]$

ii) if $0 \le x \le z_{0.05}\sigma$, quote an upper limit $u_1(x)$

iii) in order to avoid nonsensical negative upper limits, if $x < 0$, quote an upper limit of $u_1(0)$.

This is equivalent to the following rules for the ends of the confidence interval:

$$\ell(x) = \begin{cases} 0 & x \le z_{0.05}\sigma \\ \ell_2(x) & x > z_{0.05}\sigma \end{cases} \tag{3.7a}$$

$$u(x) = \begin{cases} u_1(0) & x < 0 \\ u_1(x) & 0 \le x \le z_{0.05}\sigma \\ u_2(x) & x > z_{0.05}\sigma \end{cases} \tag{3.7b}$$

On the same set of axes, plot $\ell(x)$ and $u(x)$ versus x. The confidence interval for a given measurement x is a vertical line between these two curves.

[1] Introduced in Feldman and Cousins, "Unified approach to the classical statistical analysis of small signals", *Phys. Rev. D* **57**, 3873 (1998)

c) Frequentist confidence belt constructions are defined not in terms of vertical slices of the graph you made, but horizontal ones. The *coverage* of the confidence interval at a given θ is

$$P(\ell(X) \leq \theta \leq u(X)|\theta) = \int_{u^{-1}(\theta)}^{\ell^{-1}(\theta)} f(x|\theta)\, dx \tag{3.8}$$

Calculate the coverage of the confidence intervals defined in (3.7), as a function of $\theta > 0$. You should find four different experessions corresponding to different ranges of θ. (You may find it useful to use your graph from part c) to work out the inverse functions ℓ^{-1} and u^{-1}.) If they were actual 90% intervals, this number would be 0.90 for all θ, but it is not, because of the flip-flopping. (Coverage of greater than 90% is referred to as conservative and generally considered acceptable, but undercoverage is not.)