SPACETIME COARSE GRAININGS IN RELATIVISTIC PARTICLE MOTION

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ABSTRACT
Hartle's generalized quantum mechanics formalism is used to examine a simple spacetime coarse graining, i.e., a set of alternatives defined with respect to regions extended in time as well as space, in the quantum mechanics of a free relativistic particle. For suitable initial conditions, tractable formulas are found for branch wave functions. Some initial conditions are found to give decoherence and allow the consistent assignment of probabilities.

1. Introduction

Most formulations of quantum mechanics make predictions for alternatives defined "at a moment of time", or more generally on a spacelike surface. However, in a quantum theory of gravity, in which the metric itself is behaving quantum-mechanically, the notion of "spacelike" should be ill-defined. It is constructive, then, to consider a theory which predicts spacetime alternatives, defined with respect to regions extended in time as well as space. Spacetime alternatives in nonrelativistic quantum mechanics have been considered in the past by Feynman,[1] Yamada and Takagi,[2] and Hartle.[3] (And also in this volume by Yamada.) The present work examines spacetime alternatives for the quantum mechanics of a free relativistic particle in Minkowski space, which should be a closer analogy to gravity because it exhibits a single reparametrization invariance, which can be thought of as a subset of the diffeomorphism invariance exhibited by general relativity.

We use Hartle's generalized quantum mechanics formalism[4] in which the possible histories of the system ("fine-grained histories") are partitioned into classes $\{c_\alpha\}$ ("coarse graining"). For each coarse graining, a complex matrix $D(\alpha,\alpha')$ ("decoherence functional") is constructed according to certain conditions. If its off-diagonal elements vanish to some accuracy $[D(\alpha,\alpha')\approx\delta_{\alpha\alpha'}p(\alpha)]$ ["(medium) decoherence"], quantum-mechanical interference is negligible and the diagonal elements are the probabilities of the alternatives to the same accuracy. We construct the decoherence functional by a sum over histories; the fine-grained histories over which we sum are arbitrary paths through spacetime, not just those which are single-valued in the "time" coördinate x^0. This construction is Lorentz invariant.

Here we present the decoherence functional for one simple coarse-graining which decoheres for certain initial and final conditions. For more details and a wider consideration of the subject, the reader is directed to the author's recent paper.[5]

2. Formulation[6]

For each class c_α of paths, a restricted propagator K_α is constructed via a sum over those histories which start at a spacetime point x' on an initial spacelike 3-

surface σ', end at a point x'' on a final surface σ'' and are in the class c_α:

$$K_\alpha(x'', x') = \frac{1}{2mi} \int_0^\infty dN \int_{x'\alpha x''} \mathcal{D}x\, \mathcal{D}p \exp\left[i \int_0^N d\tau \left(p \cdot \frac{dx}{d\tau} - \frac{p^2 + m^2}{2m} \right) \right]. \qquad (1)$$

The sum of all the $\{K_\alpha\}$ in a coarse graining gives the Feynman propagator: $\sum_\alpha K_\alpha(x'', x') = \Delta_F(x'' - x')$. The initial condition is described by a Klein-Gordon wave function $\psi(x')$; to each alternative c_α there corresponds a branch wave function $\psi_\alpha(x'') = iK_\alpha(x'', x') \circ \psi(x')$ where $\circ$ is the Klein-Gordon inner product on σ'. Summing all the $\{\psi_\alpha\}$ gives the positive frequency component of ψ: $\sum_\alpha \psi_\alpha(x'') = i\Delta_F(x'' - x') \circ \psi(x') = \psi^+(x'')$. The final condition is a mixed state described by a set of Klein-Gordon wave functions $\{\varphi_i(x'')\}$ and non-negative weights $\{p_i''\}$, attached with the Klein-Gordon inner product on σ'' to produce the decoherence functional:

$$D(\alpha, \alpha') = \frac{\sum_i (\psi_{\alpha'} \circ \varphi_i) p_i'' (\varphi_i \circ \psi_\alpha)}{\sum_i (\psi^+ \circ \varphi_i) p_i'' (\varphi_i \circ \psi^+)} = \frac{\psi_{\alpha'} \circ \rho'' \circ \psi_\alpha}{\psi^+ \circ \rho'' \circ \psi^+}. \qquad (2)$$

We would like to choose for ρ'' a condition of future indifference, i.e., a completely unspecified final condition. This condition is usually implemented by replacing the final density matrix with the identity operator. However, for the (non-positive-definite) Klein-Gordon inner product, the identity operator is given by

$$I(x_2, x_1) \cong \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{i\mathbf{p}\cdot(\mathbf{x}_2 - \mathbf{x}_1)} \left(e^{-i\omega_{\mathbf{p}}(t_2 - t_1)} - e^{i\omega_{\mathbf{p}}(t_2 - t_1)} \right), \qquad (3)$$

which is unacceptable as a final density matrix, since some of the weights $\{p_i''\}$ it implies are negative. Instead, we must take the final condition to be

$$\rho''(x_2, x_1) = \int \frac{d^3p}{(2\pi)^3 2\omega_{\mathbf{p}}} e^{i\mathbf{p}\cdot(\mathbf{x}_2 - \mathbf{x}_1)} \left(e^{-i\omega_{\mathbf{p}}(t_2 - t_1)} + e^{i\omega_{\mathbf{p}}(t_2 - t_1)} \right), \qquad (4)$$

so that $\psi_{\alpha'} \circ \rho'' \circ \psi_\alpha = \psi_{\alpha'}^+ \circ \psi_\alpha^+ - \psi_{\alpha'}^- \circ \psi_\alpha^- = \psi_{\alpha'} \circ \psi_\alpha - 2\psi_{\alpha'}^- \circ \psi_\alpha^-$, which differs from $\psi_{\alpha'} \circ \psi_\alpha$ whenever ψ_α and $\psi_{\alpha'}$ have negative frequency components.

3. Coarse Graining and Results

Let n be a constant spacelike unit vector ($n \cdot n = 1$), and $x_n = n \cdot x$ be the component of x along n. We can use the timelike 3-surface $x_n = 0$ to define a set of three alternatives: c_ℓ, in which the entire path lies in the region $x_n < 0$; c_r, in which the path is confined to the region $x_n > 0$; and c_b, in which the path crosses $x_n = 0$. If we define the reflection of x through the plane $x_n = 0$ by $x_c = x - 2x_n n$, it is possible to show by the method of images[5] that $K_\ell(x'', x') = \Theta(-x_n'')\,[\Delta_F(x'' - x') - \Delta_F(x'' - x_c')]\,\Theta(-x_n')$ and $K_r(x'', x') = \Theta(x_n'')\,[\Delta_F(x'' - x') - \Delta_F(x'' - x_c')]\,\Theta(x_n')$; the third restricted propagator can be found by superposition: $K_b = \Delta_F - K_\ell - K_r$. If we choose the initial state $\psi(x')$ to be antisymmetric about $x_n = 0$, (i.e., $\psi(x_c) = -\psi(x)$), the branch wave functions take the particularly simple forms $\psi_\ell(x'') = \Theta(-x_n'')\psi^+(x'')$,

$\psi_r(x'') = \Theta(x''_n)\psi^+(x'')$, and $\psi_b(x'') = 0$. If the initial state is normalized so that $\psi^+ \circ \psi^+ = 1$, the decoherence functional is then

$$\begin{pmatrix} D(\ell,\ell) = \frac{1}{2} + \Delta D & D(\ell,r) = -\Delta D & D(\ell,b) = 0 \\ D(r,\ell) = -\Delta D & D(r,r) = \frac{1}{2} + \Delta D & D(r,b) = 0 \\ D(b,\ell) = 0 & D(b,r) = 0 & D(b,b) = 0 \end{pmatrix} \tag{5}$$

where $\Delta D = -2\psi_\ell^- \circ \psi_\ell^- = -2\psi_r^- \circ \psi_r^- = 2\psi_\ell^- \circ \psi_r^- = 2\psi_r^- \circ \psi_\ell^-$ is given by[a]

$$\Delta D = 2\int \frac{dk_{1n}dk_{2n}d^2k_\perp}{(2\pi)^2} \frac{\omega_1 + \omega_2}{2\sqrt{\omega_1\omega_2}} \tilde{\psi}^+(\mathbf{k}_2)^* \tilde{\psi}^+(\mathbf{k}_1) \frac{e^{-i(\omega_1-\omega_2)t''}}{k_{1n} - k_{2n}} \ln\left(\frac{\omega_1 - k_{1n}}{\omega_2 - k_{2n}}\right). \tag{6}$$

Note that whenever the alternatives do decohere ($\Delta D \approx 0$), the probabilities are given by $p(\ell) \approx 1/2 \approx p(r)$, $p(b) = 0$.

One special initial condition is for $\tilde{\psi}^+$ to be peaked with a small width δk around a single wavenumber $\mathbf{k}$ (and its reflection $\mathbf{k}_c$). Then ΔD becomes

$$\Delta D = -4\frac{\delta k}{(2\pi)^{3/2}}\left[\frac{1}{\omega_0} - \frac{1}{k_{0n}}\ln\left(\frac{\omega_0 - k_{0n}}{\omega_{0\perp}}\right)\right] + \mathcal{O}([\delta k]^2), \tag{7}$$

and we have approximate decoherence to lowest order in δk.

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4. See, for example, J. B. Hartle, in *Quantum Cosmology and Baby Universes,* Proceedings of the 7th Jerusalem Winter School, 1989, edited by S. Coleman, J. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991).
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6. For more details, see J. B. Hartle, in *Gravitation and Quantizations,* Proceedings of the 1992 Les Houches Summer School, edited by B. Julia and J. Zinn-Justin (North Holland, Amsterdam, 1994).

[a] We use here several pieces of notation, namely $\mathbf{v}_\perp = \mathbf{v} - v_n\mathbf{n}$ and $\omega_\perp = \sqrt{\mathbf{k}_\perp^2 + m^2}$, and also that $\tilde{\psi}^+$ is the Fourier transform of the positive energy part of ψ. We are also working in a reference frame where $\mathbf{n}$ has no time component and with a final surface σ'' which is a surface of constant time t''.